The Bright Side of Mathematics

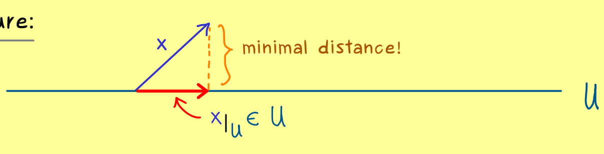
Hilbert Spaces - Part 8

Approximation Formula: $(X, \langle \cdot , \cdot \rangle)$ Hilbert space, $U \subseteq X$ ($\neq \emptyset$) closed + convex.

Then for every $x \in X$ there is a unique best approximation: $x|_U \in U$

This means: $\| x - x|_U \| = \mathrm{dist}(x, U) := \inf \{ \|x-u\| \mid u \in U \}$

Rough picture:



Proof: (1) Existence: $\inf \{ \|x-u\| \mid u \in U \}$ ← can be approximated by a sequence in U

$\Longrightarrow$ there is $(u_n)_{n \in \mathbb{N}} \subseteq U$ with $\|x-u_n\| \xrightarrow{n \to \infty} \mathrm{dist}(x,U) := \inf \{ \|x-u\| \mid u \in U \}$

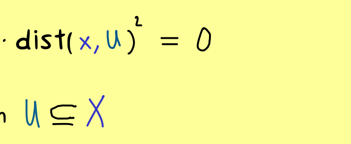
Let's calculate with this sequence:

parallelogram law

$$\|u_n - u_m\|^2 = \|\underbrace{(u_n - x)}_{v} - \underbrace{(u_m - x)}_{w}\|^2 = 2 \cdot \|v\|^2 + 2 \cdot \|w\|^2 - \|v+w\|^2$$

$$= 2 \cdot \left(\|x-u_n\|^2 + \|x-u_m\|^2 \right) - \|u_n + u_m - 2x\|^2$$

$$= 2 \cdot \left(\|x-u_n\|^2 + \|x-u_m\|^2 \right) - \left(2 \, \|x - \underbrace{\tfrac{1}{2}(u_n+u_m)}_{\text{convex combination}} \| \right)^2$$



We get for the second part:

$$2 \, \|x - \tfrac{1}{2}(u_n+u_m)\| \geq 2 \cdot \mathrm{dist}(x, U) \quad \text{and}$$

$$2 \, \|x - \tfrac{1}{2}(u_n+u_m)\| = \|2 \cdot x - u_n - u_m\| \leq \|x-u_n\| + \|x-u_m\| \xrightarrow{n,m \to \infty} 2 \cdot \mathrm{dist}(x,U)$$

We get: $\|u_n - u_m\|^2 = 2 \cdot \left(\|x-u_n\|^2 + \|x-u_m\|^2 \right) - \left(2 \, \|x - \tfrac{1}{2}(u_n+u_m)\| \right)^2$

$$\xrightarrow{n,m \to \infty} 4 \cdot \mathrm{dist}(x,U)^2 - 4 \cdot \mathrm{dist}(x,U)^2 = 0$$

$\Longrightarrow$ $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $U \subseteq X$

$\overset{X \text{ complete}}{\Longrightarrow}$ limit exists in X

$\overset{U \text{ closed}}{\Longrightarrow}$ limit lies in U $\Longrightarrow$ best approximation exists

(2) Uniqueness: Assume we have two best approximations $\tilde{u}, \hat{u} \in U$

$$\mathrm{dist}(x,U) = \|x - \tilde{u}\| = \|x - \hat{u}\|$$

Consider sequence $(u_n)_{n \in \mathbb{N}}$ given by $(\tilde{u}, \hat{u}, \tilde{u}, \hat{u}, \tilde{u}, \hat{u}, \ldots)$

It satisfies: $\|x-u_n\| \xrightarrow{n \to \infty} \mathrm{dist}(x,U)$

$\Longrightarrow$ $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence = convergent!

$\Longrightarrow$ $\tilde{u} = \hat{u}$ □