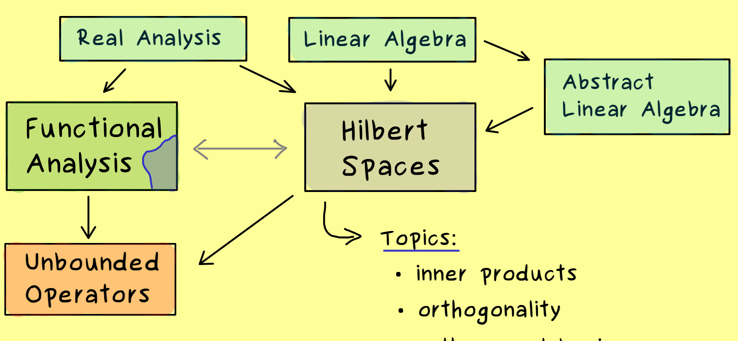
The Bright Side of Mathematics

Hilbert Spaces - Part 1



Topics:

- inner products
- orthogonality
- orthonormal basis
- orthogonal projections
- unitary + self-adjoint operators

Definition: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. An $\mathbb{F}$-vector space X with inner product $\langle \cdot , \cdot \rangle : X \times X \to \mathbb{F}$, which means

(1) $\langle x, x \rangle \geq 0$ for all $x \in X$ (positive definite)

and $\langle x, x \rangle = 0 \Rightarrow x = 0$ (zero vector)

(2) $\langle y, x + \tilde{x} \rangle = \langle y, x \rangle + \langle y, \tilde{x} \rangle$ for all $x, \tilde{x}, y \in X$

$\langle y, \lambda \cdot x \rangle = \lambda \cdot \langle y, x \rangle$ for all $\lambda \in \mathbb{F}, x, \tilde{x}, y \in X$

(linear in the second argument)

(3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$ (conjugate symmetric)

is called an inner-product space. (pre-Hilbert space)

Cauchy-Schwarz inequality: For an inner product space $(X, \langle \cdot , \cdot \rangle)$, we have:

$$|\langle y, x \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \quad \text{for all } x, y \in X$$

Proof: For $y \neq 0$:

$$0 \leq \left\langle x - \frac{\langle y, x \rangle}{\langle y, y \rangle} \cdot y \, , \, x - \frac{\langle y, x \rangle}{\langle y, y \rangle} \cdot y \right\rangle$$

$$= \langle x, x \rangle - \frac{\overline{\langle y, x \rangle}}{\langle y, y \rangle} \cdot \langle y, x \rangle - \frac{\langle y, x \rangle}{\langle y, y \rangle} \cdot \langle x, y \rangle$$

$$+ \frac{\overline{\langle y, x \rangle}}{\langle y, y \rangle} \cdot \frac{\langle y, x \rangle}{\langle y, y \rangle} \cdot \langle y, y \rangle$$

$$= \langle x, x \rangle - \frac{|\langle y, x \rangle|^2}{\langle y, y \rangle}$$

□

Result: $\|x\| := \sqrt{\langle x, x \rangle}$ defines a norm on X

Definition: An inner product space $(X, \langle \cdot , \cdot \rangle)$ is called a Hilbert space if $(X, \|\cdot\|)$ is complete.