



Functional Analysis - Part 34

Spectral theorem of compact operators

Let X be a complex Hilbert space and $T: X \rightarrow X$ be a compact operator.

Assume that T is self-adjoint ($T^* = T$) or normal ($T^*T = TT^*$).

Then there is an orthonormal system $\{e_i \mid i \in I\}$ with $I \subseteq \mathbb{N}$

and a sequence $(\lambda_i)_{i \in I}$ in $\mathbb{C} \setminus \{0\}$ with $\lambda_i \xrightarrow{i \rightarrow \infty} 0$ (if I infinite)

such that:

$$X = \text{Ker}(T) \oplus^\perp \overline{\text{Span}(e_i \mid i \in I)}$$

orthogonal sum: $X = U \oplus^\perp V$ means:

for each $x \in X$ there is $u \in U, v \in V$:

$$\bullet x = u + v$$

$$\bullet u \perp v$$

↑ ↗
unique!

and
$$Tx = \sum_{k \in I} \lambda_k e_k \langle e_k, x \rangle \quad \text{for } x \in X$$

↑ ↗
eigenvalue eigenvector to λ_k

and
$$\|T\| = \sup_{k \in I} |\lambda_k|.$$