The Bright Side of Mathematics

Functional analysis – part 19

Hölder's inequality (for $\mathbb{F}^n$ and $p \in (1,\infty)$)

↳ $p' \in (1,\infty)$ Hölder conjugate

For $x \in \mathbb{F}^n$:

$$\|x\|_q := \left(\sum_{j=1}^{n} |x_j|^q \right)^{\frac{1}{q}} , \quad q \in [1,\infty) \qquad \boxed{\frac{1}{p} + \frac{1}{p'} = 1}$$

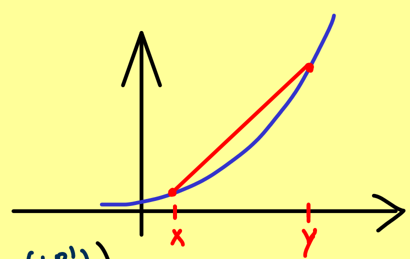
For $x, y \in \mathbb{F}^n$ write: $\quad xy := \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_n y_n \end{pmatrix}$

Then: $\|xy\|_1 \leq \|x\|_p \cdot \|y\|_{p'}$ for all $x, y \in \mathbb{F}^n$

Young's inequality:

$a, b > 0 \Rightarrow a \cdot b \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$

Proof: $f : x \mapsto e^x$ is convex: $\lambda \in [0,1]$



$$\underset{\substack{\| \\ f(\log(a) + \log(b)) \\ \| \\ a \cdot b}}{} \quad f(\underbrace{\lambda}_{\frac{1}{p}} \underbrace{x}_{\log(a^p)} + \underbrace{(1-\lambda)}_{\frac{1}{p'}} \underbrace{y}_{\log(b^{p'})}) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$= \frac{1}{p} f(\log(a^p)) + \frac{1}{p'} f(\log(b^{p'}))$$

$$= \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$$

Proof of Hölder's inequality:

1^{st} case: $x = 0$ or $y = 0$

2^{nd} case:

$$\frac{1}{\|x\|_p \cdot \|y\|_{p'}} \|xy\|_1 = \frac{1}{\|x\|_p \cdot \|y\|_{p'}} \sum_{j=1}^{n} |x_j y_j| = \sum_{j=1}^{n} \overbrace{\frac{|x_j|}{\|x\|_p}}^{a} \cdot \overbrace{\frac{|y_j|}{\|y\|_{p'}}}^{b}$$

$$\leq \sum_{j=1}^{n} \frac{1}{p} \cdot \frac{|x_j|^p}{\|x\|_p^p} + \sum_{j=1}^{n} \frac{1}{p'} \frac{|y_j|^{p'}}{\|y\|_{p'}^{p'}} = \frac{1}{p} + \frac{1}{p'} = 1$$