



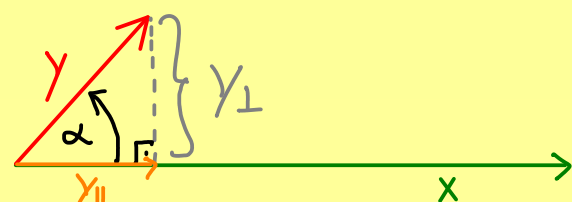
The Bright Side of Mathematics

Functional analysis - part 10

Cauchy-Schwarz inequality: Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $\|x\| := \sqrt{\langle x, x \rangle}$. Then for all $x, y \in X$:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

and $|\langle x, y \rangle| = \|x\| \cdot \|y\| \iff x, y$ linearly dependent



Proof: 1st case: $x = 0$: $|\langle x, y \rangle| = 0 = \|x\| \cdot \|y\|$ ✓

2nd case $x \neq 0$: $\hat{x} := \frac{x}{\|x\|}$, $y_{\parallel} := \langle \hat{x}, y \rangle \hat{x}$, $y_{\perp} := y - y_{\parallel}$

$$\begin{aligned} 0 \leq \|y_{\perp}\|^2 &= \|y - y_{\parallel}\|^2 = \|y - \langle \hat{x}, y \rangle \hat{x}\|^2 = \langle y - \langle \hat{x}, y \rangle \hat{x}, y - \langle \hat{x}, y \rangle \hat{x} \rangle \\ &= \langle y - \langle \hat{x}, y \rangle \hat{x}, y \rangle - \langle y - \langle \hat{x}, y \rangle \hat{x}, \langle \hat{x}, y \rangle \hat{x} \rangle \\ &= \langle y, y \rangle - \langle \langle \hat{x}, y \rangle \hat{x}, y \rangle - \langle y, \langle \hat{x}, y \rangle \hat{x} \rangle + \langle \langle \hat{x}, y \rangle \hat{x}, \langle \hat{x}, y \rangle \hat{x} \rangle \\ &= \|y\|^2 - \left(\langle \langle \hat{x}, y \rangle \hat{x}, y \rangle + \overline{\langle \langle \hat{x}, y \rangle \hat{x}, y \rangle} \right) + \|\langle \hat{x}, y \rangle \hat{x}\|^2 \\ &= \|y\|^2 - \left(2 \cdot \operatorname{Re}(\langle \langle \hat{x}, y \rangle \hat{x}, y \rangle) \right) + |\langle \hat{x}, y \rangle|^2 \cdot \underbrace{\|\hat{x}\|^2}_{=1} \\ & \quad \frac{\langle \hat{x}, y \rangle \langle \hat{x}, y \rangle}{\langle \hat{x}, y \rangle \langle \hat{x}, y \rangle} = |\langle \hat{x}, y \rangle|^2 \\ &= \|y\|^2 - 2|\langle \hat{x}, y \rangle|^2 + |\langle \hat{x}, y \rangle|^2 = \underline{\|y\|^2 - |\langle \hat{x}, y \rangle|^2} \end{aligned}$$

$$\Rightarrow \|y\|^2 \geq |\langle \hat{x}, y \rangle|^2 = \left| \left\langle \frac{x}{\|x\|}, y \right\rangle \right|^2 = \frac{1}{\|x\|^2} |\langle x, y \rangle|^2$$

$$\Rightarrow \|x\| \cdot \|y\| \geq |\langle x, y \rangle|$$

Δ -inequality for $\|\cdot\|$: $\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + 2 \operatorname{Re}(\langle x, y \rangle) + \|y\|^2$

$$\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2$$

$$\stackrel{\text{Cauchy Schwarz}}{\leq} \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$