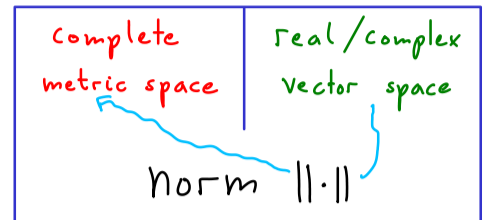


## Functional analysis - part 7

Banach space:



Examples: (1)  $\mathbb{R}$  is a one-dimensional real vector space

$|\cdot|: \mathbb{R} \rightarrow [0, \infty)$  is a norm.  $d_{1,1}(x, y) := |x - y|$  is a metric.

$\Rightarrow (\mathbb{R}, |\cdot|)$  is a Banach space

(2)  $X = \{0\}$ , zero-dimensional real vector space.

$\|\cdot\|: X \rightarrow [0, \infty)$  defined by  $\|0\| := 0$ .

$\Rightarrow (X, \|\cdot\|)$  is a Banach space.

(3) Let  $\ell^p(\mathbb{N}, \mathbb{F})$  (where  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ ,  $p \in [1, \infty)$ )

be defined as all sequences  $x = (x_n)_{n \in \mathbb{N}}$  in  $\mathbb{F}$  such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty \quad (\text{converges!})$$

Then  $\|\cdot\|_p: \ell^p \rightarrow [0, \infty)$  with  $\|x\|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$  is a norm!  
(Show later!)

Claim:  $(\ell^p, \|\cdot\|_p)$  is a Banach space

Proof: •  $\ell^p$  is an  $\mathbb{F}$ -vector space and  $\|\cdot\|_p$  is a norm on it (see later).

• Completeness: Let  $(x^{(k)})_{k \in \mathbb{N}}$  be a Cauchy sequence in  $\ell^p$ .

$$\begin{array}{l} x^{(1)} = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_5^{(1)}, x_6^{(1)}, x_7^{(1)}, x_8^{(1)} \dots) \\ x^{(2)} = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}, x_5^{(2)}, x_6^{(2)}, x_7^{(2)}, x_8^{(2)} \dots) \\ x^{(3)} = (x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, x_5^{(3)}, x_6^{(3)}, x_7^{(3)}, x_8^{(3)} \dots) \\ \vdots \\ \tilde{x} := (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8 \dots) \end{array}$$

$$|x_m^{(k)} - x_m^{(l)}|^p \leq \sum_{n=1}^{\infty} |x_n^{(k)} - x_n^{(l)}|^p = \|x^{(k)} - x^{(l)}\|_p^p$$

Cauchy sequence:  $\forall \varepsilon > 0 \exists K \in \mathbb{N} \forall k, l \geq K : \|x^{(k)} - x^{(l)}\|_p < \varepsilon$

$$|x_m^{(k)} - x_m^{(l)}| \leq \varepsilon \Rightarrow (x_m^{(k)})_{k \in \mathbb{N}} \text{ Cauchy seq. in } \mathbb{F}$$

$\Rightarrow (x_m^{(k)})_{k \in \mathbb{N}}$  has a limit  $\tilde{x}_m \in \mathbb{F}$

Let  $\varepsilon > 0$ , choose  $K \in \mathbb{N}$  such that  $\forall k, l \geq K : \|x^{(k)} - x^{(l)}\|_p < \varepsilon' =: \frac{\varepsilon}{2}$

$$\|x^{(k)} - \tilde{x}\|_p^p = \sum_{n=1}^{\infty} |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N \rightarrow \infty} \sum_{n=1}^N |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N \rightarrow \infty} \lim_{l \rightarrow \infty} \sum_{n=1}^N |x_n^{(k)} - x_n^{(l)}|^p < (\varepsilon')^p$$

Then for all  $k \geq K$ :  $\|x^{(k)} - \tilde{x}\|_p \leq (\varepsilon') < \varepsilon$

And  $\tilde{x} = \underbrace{\tilde{x} - x^{(k)}}_{\in \ell^p} + \underbrace{x^{(k)}}_{\in \ell^p} \in \ell^p$  (it's a vector space!)