Functional analysis - part 7

Banach space: | complete metric space | real/complex vector space |
norm $\|\cdot\|$

Examples: (1) $\mathbb{R}$ is a one-dimensional real vector space

$|\cdot| : \mathbb{R} \longrightarrow [0,\infty)$ is a norm. $d_{|\cdot|}(x,y) := |x-y|$ is a metric.

$\Longrightarrow (\mathbb{R}, |\cdot|)$ is a Banach space

(2) $X = \{0\}$, zero-dimensional real vector space.

$\|\cdot\| : X \longrightarrow [0,\infty)$ defined by $\|0\| := 0$.

$\Rightarrow (X, \|\cdot\|)$ is a Banach space.

(3) Let $\overset{\ell^p \,=}{\ell^p}(\mathbb{N}, \mathbb{F})$ (where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $p \in [1,\infty)$)

be defined as all sequences $x = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty \qquad \text{(converges!)}$$

Then $\|\cdot\|_p : \ell^p \longrightarrow [0,\infty)$ with $\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$ is a $\underbrace{\text{norm}}_{\text{(Show later!)}}$!

Claim: $(\ell^p, \|\cdot\|_p)$ is a Banach space

Proof: • ℓ^p is an $\mathbb{F}$-vector space and $\|\cdot\|_p$ is a norm on it (see later).

• Completeness: Let $(x^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in ℓ^p.

$$
\begin{aligned}
x^{(1)} &= (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \boxed{x_4^{(1)}}, x_5^{(1)}, x_6^{(1)}, x_7^{(1)}, x_8^{(1)} \dots) \\
x^{(2)} &= (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}, x_5^{(2)}, x_6^{(2)}, x_7^{(2)}, x_8^{(2)} \dots) \\
x^{(3)} &= (x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, x_5^{(3)}, x_6^{(3)}, x_7^{(3)}, x_8^{(3)} \dots) \\
&\ \ \vdots \\
? \quad \tilde{x} &:= (\tilde{x}_1\ \ \tilde{x}_2\ \ \tilde{x}_3\ \ \tilde{x}_4\ \ \tilde{x}_5\ \ \tilde{x}_6\ \ \tilde{x}_7\ \ \tilde{x}_8 \dots)
\end{aligned}
$$

$$|x_m^{(k)} - x_m^{(\ell)}|^p \leq \sum_{n=1}^{\infty} |x_n^{(k)} - x_n^{(\ell)}|^p = \|x^{(k)} - x^{(\ell)}\|_p^p$$

Cauchy sequence: $\forall \varepsilon > 0 \ \exists K \in \mathbb{N} \ \forall k, \ell \geq K : \|x^{(k)} - x^{(\ell)}\|_p < \varepsilon$

$|x_m^{(k)} - x_m^{(\ell)}| \leq$ $\Rightarrow$ $(x_m^{(k)})_{k\in\mathbb{N}}$ Cauchy seq. in $\mathbb{F}$

$\Longrightarrow$ $(x_m^{(k)})_{k\in\mathbb{N}}$ has a limit $\tilde{x}_m \in \mathbb{F}$

Let $\varepsilon > 0$, choose $K \in \mathbb{N}$ such that $\forall k, \ell \geq K$: $\|x^{(k)} - x^{(\ell)}\|_p < \varepsilon' =: \frac{\varepsilon}{2}$

$$\|x^{(k)} - \tilde{x}\|_p^p = \sum_{n=1}^{\infty} |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N\to\infty} \sum_{n=1}^{N} |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N\to\infty} \lim_{\ell\to\infty} \underbrace{\sum_{n=1}^{N} |x_n^{(k)} - x_n^{(\ell)}|^p}_{<(\varepsilon')^p}$$

Then for all $k \geq K$: $\|x^{(k)} - \tilde{x}\|_p \leq (\varepsilon') < \varepsilon$

And $\tilde{x} = \underbrace{\tilde{x} - x^{(k)}}_{\in \ell^p} + \underbrace{x^{(k)}}_{\in \ell^p} \in \ell^p$ (it's a vector space!)