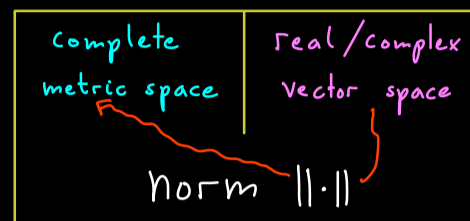


Functional analysis - part 7

Banach space:



Examples: (1) \mathbb{R} is a one-dimensional real vector space
 $|\cdot|: \mathbb{R} \rightarrow [0, \infty)$ is a norm. $d_{|\cdot|}(x, y) := |x - y|$ is a metric.
 $\Rightarrow (\mathbb{R}, |\cdot|)$ is a Banach space

(2) $X = \{0\}$, zero-dimensional real vector space.
 $\|\cdot\|: X \rightarrow [0, \infty)$ defined by $\|0\| := 0$.
 $\Rightarrow (X, \|\cdot\|)$ is a Banach space.

(3) Let $\ell^p(\mathbb{N}, \mathbb{F})$ (where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $p \in [1, \infty)$)
be defined as all sequences $x = (x_n)_{n \in \mathbb{N}}$ in \mathbb{F} such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty \quad (\text{converges!})$$

Then $\|\cdot\|_p: \ell^p \rightarrow [0, \infty)$ with $\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$ is a norm!
(Show later!)

Claim: $(\ell^p, \|\cdot\|_p)$ is a Banach space

Proof: • ℓ^p is an \mathbb{F} -vector space and $\|\cdot\|_p$ is a norm on it (see later).

• Completeness: Let $(x^{(k)})_{k \in \mathbb{N}}$ be a Cauchy sequence in ℓ^p .

$$\begin{aligned} x^{(1)} &= (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_5^{(1)}, x_6^{(1)}, x_7^{(1)}, x_8^{(1)} \dots) \\ x^{(2)} &= (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}, x_5^{(2)}, x_6^{(2)}, x_7^{(2)}, x_8^{(2)} \dots) \\ x^{(3)} &= (x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, x_5^{(3)}, x_6^{(3)}, x_7^{(3)}, x_8^{(3)} \dots) \\ \vdots & \\ \tilde{x} &:= (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8 \dots) \end{aligned}$$

$$|x_m^{(k)} - x_m^{(l)}|^p \leq \sum_{n=1}^{\infty} |x_n^{(k)} - x_n^{(l)}|^p = \|x^{(k)} - x^{(l)}\|_p^p$$

Cauchy sequence: $\forall \varepsilon > 0 \exists K \in \mathbb{N} \forall k, l \geq K : \|x^{(k)} - x^{(l)}\|_p < \varepsilon$

$|x_m^{(k)} - x_m^{(l)}| \leq \varepsilon \Rightarrow (x_m^{(k)})_{k \in \mathbb{N}}$ Cauchy seq. in \mathbb{F}

$\Rightarrow (x_m^{(k)})_{k \in \mathbb{N}}$ has a limit $\tilde{x}_m \in \mathbb{F}$

Let $\varepsilon > 0$, choose $K \in \mathbb{N}$ such that $\forall k, l \geq K : \|x^{(k)} - x^{(l)}\|_p < \varepsilon' =: \frac{\varepsilon}{2}$

$$\|x^{(k)} - \tilde{x}\|_p^p = \sum_{n=1}^{\infty} |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N \rightarrow \infty} \sum_{n=1}^N |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N \rightarrow \infty} \lim_{l \rightarrow \infty} \sum_{n=1}^N |x_n^{(k)} - x_n^{(l)}|^p$$

$< (\varepsilon')^p$

Then for all $k \geq K$: $\|x^{(k)} - \tilde{x}\|_p \leq (\varepsilon') < \varepsilon$

And $\tilde{x} = \underbrace{\tilde{x} - x^{(k)}}_{\in \ell^p} + \underbrace{x^{(k)}}_{\in \ell^p} \in \ell^p$ (it's a vector space!)