Functional analysis – part 29

Let X be a complex Banach space and
$T: X \longrightarrow X$ be a bounded linear operator.

$$\lambda \in \sigma(T) \iff (T - \lambda) \quad \text{not invertible}$$

Finite-dimensional example: $X = \mathbb{C}^n$, $Tx = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix}$

$$\Rightarrow \sigma(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \sigma_p(T)$$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors

Infinite-dimensional example: $X = \ell^p(\mathbb{N})$, $p \in [1, \infty)$

$$Tx = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \end{pmatrix}$$

Formally: For $\lambda_1, \lambda_2, \dots \in \mathbb{C}$ with $\sup_{j \in \mathbb{N}} |\lambda_j| < \infty$, define: $T: \ell^p(\mathbb{N}) \longrightarrow \ell^p(\mathbb{N})$
$(Tx)_j := \lambda_j x_j$

- $e_1 = (1, 0, 0, \dots)$ is an eigenvector with eigenvalue λ_1
- $e_2 = (0, 1, 0, \dots)$ is an eigenvector with eigenvalue λ_2
- $\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$\Rightarrow \sigma(T) \supseteq \{\lambda_1, \lambda_2, \dots\} = \sigma_p(T)$$

Let $\mu \in \mathbb{C}$ be a number with $\mu \notin \{\lambda_1, \lambda_2, \dots\}$ but $\mu \in \overline{\{\lambda_1, \lambda_2, \dots\}}$.

e.g. $\lambda_j = \frac{1}{j}$ then $\mu = 0$

$\Longrightarrow$ $T - \mu$ is injective

Show: $T - \mu$ is not surjective

Proof: Assume $T - \mu$ is surjective $\Rightarrow$ $T - \mu$ is bijective $\overset{\text{bounded inverse theorem}}{\Longrightarrow}$ $(T-\mu)^{-1}$ bounded

$$\Rightarrow \|(T-\mu)^{-1}\| \geq \|(T-\mu)^{-1} e_j\|_{\ell^p(\mathbb{N})} = \|(\lambda_j - \mu)^{-1} e_j\|_{\ell^p(\mathbb{N})} = |(\lambda_j - \mu)^{-1}|$$

$$= \frac{1}{|\lambda_j - \mu|} \xrightarrow{\text{for a subsequence}} \infty \quad \lightning$$

Result:

$$\sigma(T) = \underbrace{\{\lambda_1, \lambda_2, \dots\}}_{\sigma_p(T)} \cup \underbrace{\{\mu \in \mathbb{C} \mid \mu \notin \{\lambda_1, \lambda_2, \dots\} \wedge \mu \in \overline{\{\lambda_1, \lambda_2, \dots\}}\}}_{\sigma_c(T) \cancel{\cup \sigma_r(T)}}$$

$p \in [1, \infty)$