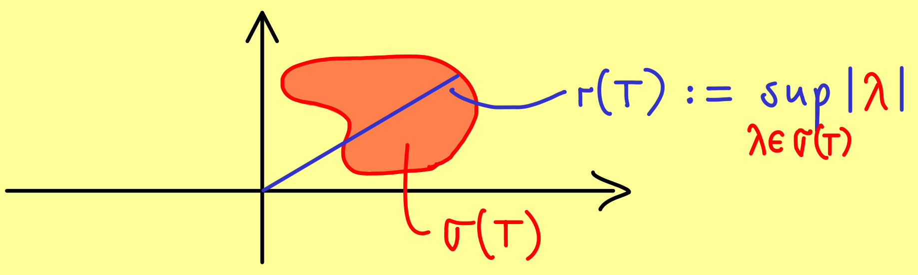
The Bright Side of Mathematics

Functional analysis – part 31

Spectral radius: X complex Banach space $\quad T: X \to X$ bounded linear operator



Theorem: X complex Banach space, $T: X \to X$ bounded linear operator.

Then: (a) $\sigma(T) \subseteq \mathbb{C}$ is compact

(b) $X \neq \{0\} \Rightarrow \sigma(T) \neq \emptyset$

(c) $r(T) := \sup\limits_{\lambda \in \sigma(T)} |\lambda| = \lim\limits_{k \to \infty} \|T^k\|^{\frac{1}{k}} = \inf\limits_{k \in \mathbb{N}} \|T^k\|^{\frac{1}{k}} \leq \|T\| < \infty$

Proof: For $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$: $\left(I - \frac{T}{\lambda}\right)^{-1} = \sum\limits_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$

$$\Rightarrow \boxed{(T - \lambda)^{-1} = -\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k \quad (*)}$$

$$\Rightarrow \sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\| \Rightarrow \sigma(T) \text{ is bounded}$$

For (b): Assume $\sigma(T) = \emptyset \Rightarrow \rho(T) = \mathbb{C}$

Reminder: The map $\rho(T) \longrightarrow \mathcal{B}(X)$

$\lambda \longmapsto (T - \lambda)^{-1}$ is analytic.

Take any $\ell \in \mathcal{B}(X)'$: $\quad f_\ell : \mathbb{C} \to \mathbb{C}$

$\lambda \mapsto \ell\left((T - \lambda)^{-1}\right)$

analytic function (holomorphic function)

We get that f_ℓ is a bounded entire function.

For $|\lambda| \geq 2 \cdot \|T\|$: $\quad \boxed{(T-\lambda)^{-1} = -\frac{1}{\lambda}\sum\limits_{k=0}^{\infty}\left(\frac{T}{\lambda}\right)^k \quad (*)}$

$$|f_\ell(\lambda)| \leq \|\ell\| \cdot \|(T - \lambda)^{-1}\| \leq \|\ell\| \underbrace{\frac{1}{|\lambda|}}_{\leq \frac{1}{2\|T\|}} \sum_{k=0}^{\infty} \underbrace{\left\|\frac{T}{\lambda}\right\|}_{\leq \frac{1}{2}}^k$$

$$\leq \frac{\|\ell\|}{\|T\|}$$

$$\xRightarrow{\text{Liouville's theorem}} f_\ell \text{ is constant}$$

$$f_\ell(0) = \ell(T^{-1})$$

$$\|$$

$$f_\ell(\lambda) = \ell\left((T - \lambda)^{-1}\right) = \ell\left(\sum_{k=0}^{\infty} (T)^{-(k+1)} \cdot (\lambda)^k\right)$$

$$= \sum_{k=0}^{\infty} \ell\left(T^{-(k+1)}\right) \cdot \lambda^k$$

$$\Rightarrow \ell(T^{-2}) = 0 \quad \text{for all } \ell \in \mathcal{B}(X)'$$

$$\xRightarrow[\text{(part 25)}]{\text{Hahn-Banach theorem}} T^{-2} = 0 \Rightarrow X = \{0\}$$