The Bright Side of Mathematics

Functional analysis - part 8

- metric $\longrightarrow$ measures distances
- norm $\longrightarrow$ measures distances, lengths
- inner product $\longrightarrow$ measures distances, lengths, angles

"$\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos(\alpha)$"

Definition: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let X be an $\mathbb{F}$-vector space.

A map $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{F}$ is called an inner product on X if

(1) $\langle x, x \rangle \geq 0$ for all $x \in X$ and $\langle x, x \rangle = 0 \iff x = 0$ [positive definite]

(2) $\langle x, y \rangle = \langle y, x \rangle$ for $\mathbb{F} = \mathbb{R}$
$\langle x, y \rangle = \overline{\langle y, x \rangle}$ for $\mathbb{F} = \mathbb{C}$
for all $x, y \in X$ [(conjugate) symmetric]

(3) $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$ for all $x, y_1, y_2 \in X$
$\langle x, \lambda \cdot y \rangle = \lambda \cdot \langle x, y \rangle$ for all $x, y \in X$, $\lambda \in \mathbb{F}$ [linear in the 2^{nd} argument]

If $\langle \cdot, \cdot \rangle$ is an inner product, then $\|x\|_{\langle \cdot,\cdot \rangle} := \sqrt{\langle x, x \rangle}$ defines norm.

Definition: $(X, \langle \cdot, \cdot \rangle)$ is called a Hilbert space if $(X, \|\cdot\|_{\langle \cdot,\cdot \rangle})$ is a Banach space.