**The Bright Side of Mathematics**

The following pages cover the whole Functional Analysis course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

The Bright Side of
**Mathematics**

# Functional analysis - part 1

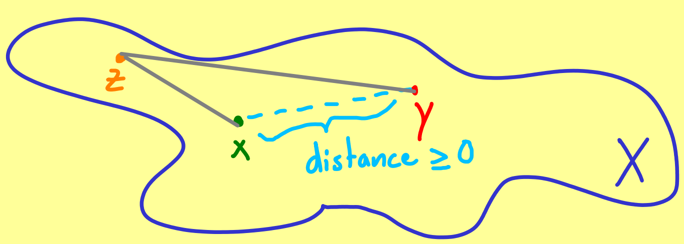
| Linear algebra ($\dim = \infty$) | + | Real and complex analysis |
|---|---|---|

$\underbrace{\qquad\qquad\qquad\qquad}$

Functional analysis
( function spaces, sequences, ... )
= Study of topological-algebraic structures

## Metric spaces

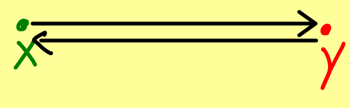
$X$ set




a metric: $d : X \times X \longrightarrow [0, \infty)$

(1) $d(x,y) = 0 \iff x = y$

(2) $d(x,y) = d(y,x)$




(3) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

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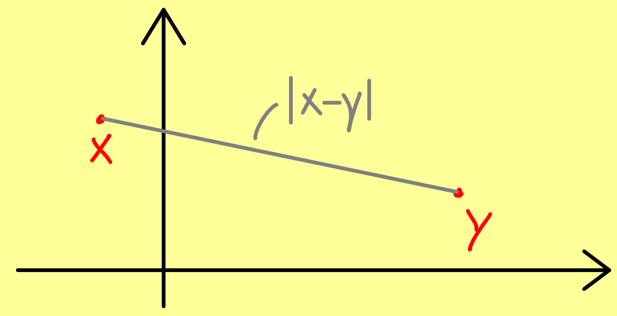
# Functional analysis – part 2

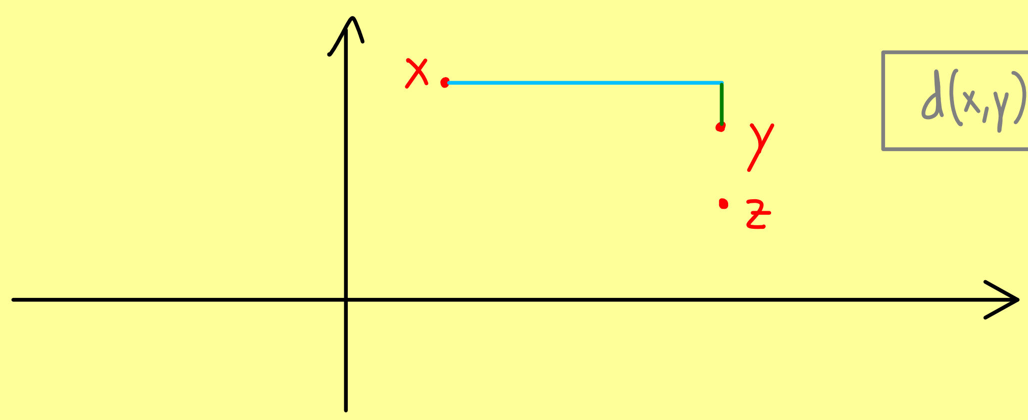
$X$ set $+$ $d: X \times X \longrightarrow [0,\infty)$ metric $=$ metric space $(X, d)$

**Examples:** (a) $X = \mathbb{C}$, $d(x,y) = |x-y|$




(b) $X = \mathbb{R}^n$, $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$ (Euclidean metric)

(c) $X = \mathbb{R}^n$, $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|, \ldots, |x_n-y_n|\}$




$\boxed{d(x,y) = d(x,z)}$

(d) $X$ any set $(\neq \emptyset)$, $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ $\boxed{\text{discrete metric}}$

$d$ is a metric: (1) ✓, (2) ✓, (3) $\Delta$-inequality: $x, y, z \in X$

First case: $x = y$: $d(x,y) = 0 \leq d(x,z) + d(z,y)$ ✓

Second case: $x \neq y$: $d(x,y) = 1 = \left\{\begin{matrix} d(x,z) \\ \text{or} \\ d(z,y) \end{matrix}\right\} \leq d(x,z) + d(z,y)$ ✓

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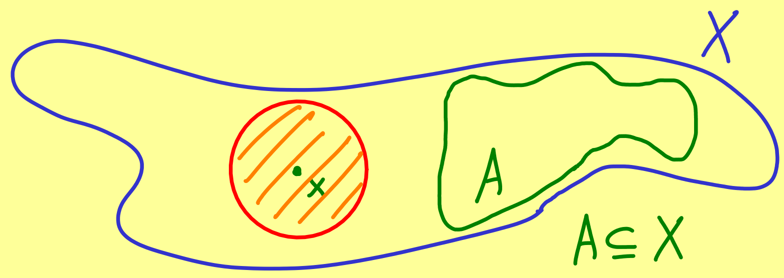
# Functional analysis – part 3

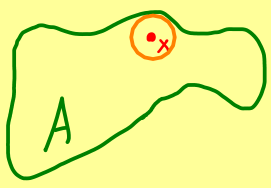
$(X, d)$ metric space




$B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$ (open ball of radius $\varepsilon > 0$ centered at $x$)

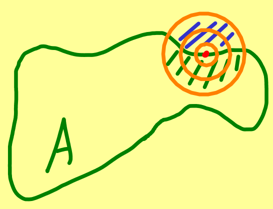
**Notions:**

(1) Open sets:




$A \subseteq X$ is called open if for each $x \in A$ there is an open ball with $B_\varepsilon(x) \subseteq A$.

(2) Boundary points:

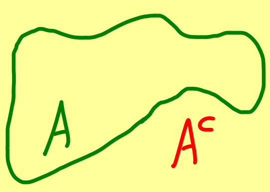



$A \subseteq X$. $x \in X$ is called a boundary point for $A$ if for all $\varepsilon > 0$: $B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$ $[A^c := X \setminus A]$

$$\partial A := \{ x \in X \mid x \text{ is boundary point for } A \}$$

Remember: $A$ open $\Leftrightarrow$ $A \cap \partial A = \emptyset$

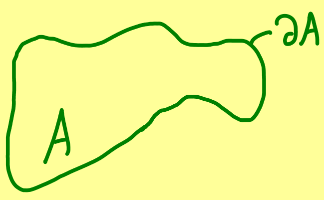
(3) Closed sets:




$A \subseteq X$ is called closed if $A^c := X \setminus A$ is open.

Remember: $A$ closed $\Leftrightarrow$ $A \cup \partial A = A$

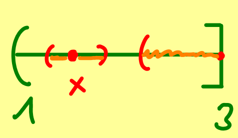
(4) Closure:




$\overline{A} := A \cup \partial A$ (always closed!)

**Example:** $X := (1, 3] \cup (4, \infty)$, $d(x,y) := |x-y|$, $(X,d)$ is a metric space

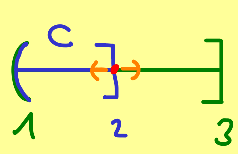
(a) $A := (1,3] \subseteq X$ open?




For $x \in A$, $x \neq 3$, define $\varepsilon := \frac{1}{2}\min(|1-x|, |3-x|)$. Then $B_\varepsilon(x) \subseteq A$.
For $x = 3$: $B_1(x) = \{ y \in X \mid d(x,y) < 1 \} = (2,3] \subseteq A$
$\Rightarrow$ $A$ is open

(b) $A$ is also closed!

(c) $C := (1,2]$, $\partial C = \{2\}$, $\overline{C} = C$



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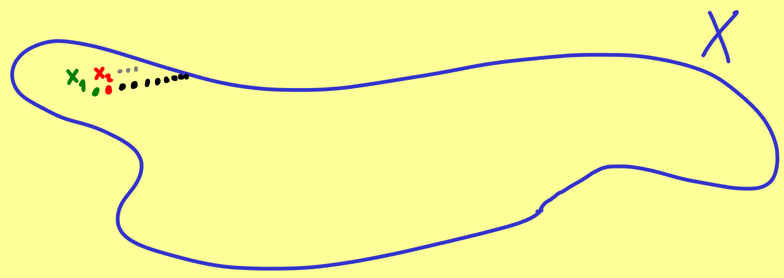
# Functional analysis - part 4

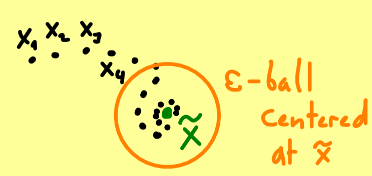
$(X, d)$ metric space




**Sequence in $X$:** $(x_1, x_2, x_3, \dots)$ or $(x_n)_{n \in \mathbb{N}}$ or $x: \mathbb{N} \to X$, $n \mapsto x_n$ map

**Convergence:** A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space $(X,d)$ is called convergent if there is $\tilde{x} \in X$ with

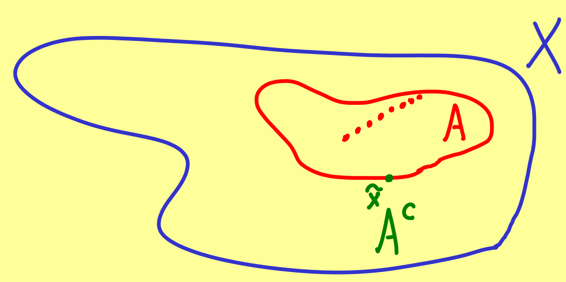
$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N : \quad d(x_n, \tilde{x}) < \varepsilon .$$




We write: $x_n \xrightarrow{n \to \infty} \tilde{x}$ or $\lim_{n\to\infty} x_n = \tilde{x}$ .

**Proposition:** $A \subseteq X$ is closed

$\Longleftrightarrow$ For every convergent sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$, one has $\lim_{n\to\infty} a_n \in A$




**Proof:** $(\Leftarrow)$: Show it by contraposition! Assume $A$ is not closed.

$\Rightarrow$ $A^c := X \setminus A$ is not open.

$\Rightarrow$ There is an $\tilde{x} \in A^c$ with $B_\varepsilon(\tilde{x}) \cap A \neq \emptyset$ for all $\varepsilon > 0$.

$\Rightarrow$ There is a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in B_{\frac{1}{n}}(\tilde{x}) \cap A$

$\Rightarrow$ $\lim_{n\to\infty} a_n = \tilde{x} \notin A$

$(\Rightarrow)$: Show it by contraposition! Assume there is $(a_n)_{n\in\mathbb{N}} \subseteq A$ with $\tilde{x} := \lim_{n\to\infty} a_n \notin A$.

$\Rightarrow$ $B_\varepsilon(\tilde{x}) \cap A \neq \emptyset$ for all $\varepsilon > 0$. $\Rightarrow$ $A^c$ is not open $\Rightarrow$ $A$ is not closed

# Functional analysis - part 5

**Example:** $X = (0,3)$ with $d(x,y) = |x-y|$ 

$(0,3)$ is closed:

- complement $\emptyset$ is open
- each convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq (0,3)$ (with limit $\tilde{x} \in X$) satisfies $\tilde{x} \in (0,3)$

What is about the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ ?

- sequence in $X$
- $d(x_n, x_m) \xrightarrow{n,m\to\infty} 0$
- it does <u>not</u> converge $\Rightarrow$ $(X,d)$ is not complete

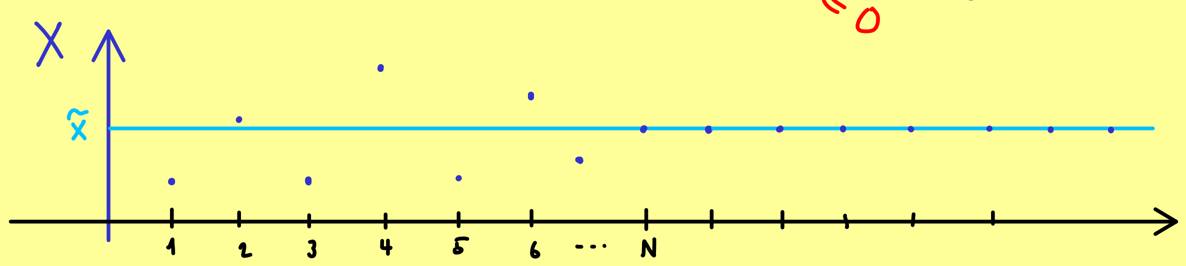
**Definition:** Let $(X,d)$ be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called <u>Cauchy sequence</u> if $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n,m \geq N : \; d(x_n, x_m) < \varepsilon$.

$(X,d)$ is called <u>complete</u> if all Cauchy sequences converge.

**Example:** (a) $X = [0,3]$ with $d(x,y) = |x-y|$ is complete.

(b) $X = (0,3)$ with $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ is complete.

Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. Take $\varepsilon = \frac{1}{2}$. Then there is an $N \in \mathbb{N}$ such that for all $n,m \geq N$, we have $\underbrace{d(x_n, x_m)}_{=0} < \frac{1}{2}$. Hence $x_n = x_m$.



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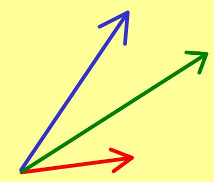
# Functional analysis - part 6

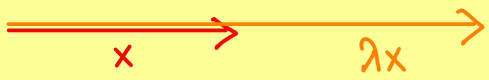


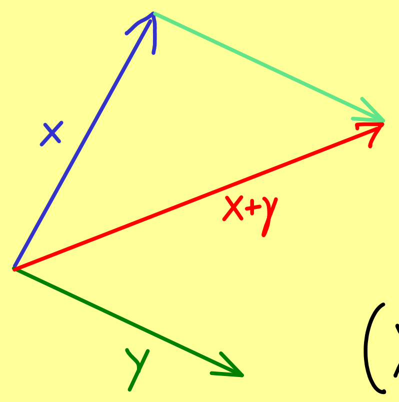
**Definition:** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $X$ be a $\mathbb{F}$-vector space.
A map $\|\cdot\| : X \longrightarrow [0, \infty)$ is called norm if

(a) $\|x\| = 0 \iff x = 0$ (positive definite)

(b) $\|\lambda \cdot x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{F}$, $x \in X$ (absolutely homogeneous)

$\nwarrow$ absolute value in $\mathbb{R}$ or $\mathbb{C}$



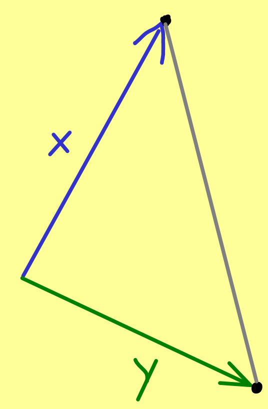

(c) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality)

$(X, \|\cdot\|)$ is then called a normed space.

**Important:** If $\|\cdot\|$ is a norm for the $\mathbb{F}$-vector space $X$, then

$$d_{\|\cdot\|}(x,y) := \|x-y\| \quad \text{defines}$$

a metric for the set $X$.




If $(X, d_{\|\cdot\|})$ is a complete metric space,
then the normed space $(X, \|\cdot\|)$ is called a Banach space.

Banach space:

| complete metric space | real/complex vector space |
|---|---|
| norm $\|\cdot\|$ | |

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# Functional analysis - part 7

Banach space: complete metric space | real/complex vector space, norm $\|\cdot\|$

**Examples:** (1) $\mathbb{R}$ is a one-dimensional real vector space

$|\cdot| : \mathbb{R} \longrightarrow [0,\infty)$ is a norm. $d_{|\cdot|}(x,y) := |x-y|$ is a metric.

$\Longrightarrow (\mathbb{R}, |\cdot|)$ is a Banach space

(2) $X = \{0\}$, zero-dimensional real vector space.

$\|\cdot\| : X \longrightarrow [0,\infty)$ defined by $\|0\| := 0$.

$\Longrightarrow (X, \|\cdot\|)$ is a Banach space.

(3) Let $\underbrace{\ell^p(\mathbb{N},\mathbb{F})}_{\ell^p}$ (where $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, $p \in [1,\infty)$)

be defined as all sequences $\underbrace{(x_n)_{n\in\mathbb{N}}}_{x}$ in $\mathbb{F}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty \qquad \text{(converges!)}$$

Then $\|\cdot\|_p : \ell^p \longrightarrow [0,\infty)$ with $\|x\|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$ $\underbrace{\text{is a norm}}_{\text{(show later!)}}$!

**Claim:** $(\ell^p, \|\cdot\|_p)$ is a Banach space

**Proof:**
- $\ell^p$ is an $\mathbb{F}$-vector space and $\|\cdot\|_p$ is a norm on it (see later).
- Completeness: Let $(x^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in $\ell^p$.

$$\begin{aligned}
x^{(1)} &= (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \boxed{x_4^{(1)}}, x_5^{(1)}, x_6^{(1)}, x_7^{(1)}, x_8^{(1)} \dots) \\
x^{(2)} &= (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}, x_5^{(2)}, x_6^{(2)}, x_7^{(2)}, x_8^{(2)} \dots) \\
x^{(3)} &= (x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, x_5^{(3)}, x_6^{(3)}, x_7^{(3)}, x_8^{(3)} \dots) \\
&\ \ \vdots \qquad ? \\
\tilde{x} &:= (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8 \dots)
\end{aligned}$$

$$|x_m^{(k)} - x_m^{(\ell)}|^p \le \sum_{n=1}^{\infty} |x_n^{(k)} - x_n^{(\ell)}|^p = \|x^{(k)} - x^{(\ell)}\|_p^p$$

Cauchy sequence: $\forall \varepsilon > 0 \ \exists K \in \mathbb{N} \ \forall k,\ell \ge K : \ \|x^{(k)} - x^{(\ell)}\|_p < \varepsilon$

$|x_m^{(k)} - x_m^{(\ell)}| \le$ $\quad \Longrightarrow (x_m^{(k)})_{k\in\mathbb{N}}$ Cauchy seq. in $\mathbb{F}$

$\Longrightarrow (x_m^{(k)})_{k\in\mathbb{N}}$ has a limit $\tilde{x}_m \in \mathbb{F}$

Let $\varepsilon > 0$, choose $K \in \mathbb{N}$ such that $\forall k,\ell \ge K$ : $\|x^{(k)} - x^{(\ell)}\|_p < \varepsilon' =: \frac{\varepsilon}{2}$

$$\|x^{(k)} - \tilde{x}\|_p^p = \sum_{n=1}^{\infty} |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N\to\infty} \sum_{n=1}^{N} |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N\to\infty} \lim_{\ell\to\infty} \underbrace{\sum_{n=1}^{N} |x_n^{(k)} - x_n^{(\ell)}|^p}_{<(\varepsilon')^p}$$

Then for all $k \ge K$: $\|x^{(k)} - \tilde{x}\|_p \le (\varepsilon') < \varepsilon$

And $\tilde{x} = \underbrace{\tilde{x} - x^{(k)}}_{\in \ell^p} + \underbrace{x^{(k)}}_{\in \ell^p} \in \ell^p$ (it's a vector space!)

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# Functional analysis - part 8

- metric $\longrightarrow$ measures distances
- norm $\longrightarrow$ measures distances, lengths
- inner product $\longrightarrow$ measures distances, lengths, angles

"$\langle x, y\rangle = \|x\| \cdot \|y\| \cdot \cos(\alpha)$"


**Definition:** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $X$ be an $\mathbb{F}$-vector space.

A map $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{F}$ is called an inner product on $X$ if

(1) $\langle x, x\rangle \geq 0$ for all $x \in X$ and $\langle x, x\rangle = 0 \iff x = 0$ [positive definite]

(2) $\langle x, y\rangle = \langle y, x\rangle$ for $\mathbb{F} = \mathbb{R}$

$\langle x, y\rangle = \overline{\langle y, x\rangle}$ for $\mathbb{F} = \mathbb{C}$

for all $x, y \in X$ [(conjugate) symmetric]

(3) $\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle$ for all $x, y_1, y_2 \in X$

$\langle x, \lambda \cdot y\rangle = \lambda \cdot \langle x, y\rangle$ for all $x, y \in X$, $\lambda \in \mathbb{F}$

[linear in the $2^{nd}$ argument]

If $\langle \cdot, \cdot \rangle$ is an inner product, then $\|x\|_{\langle \cdot, \cdot \rangle} := \sqrt{\langle x, x\rangle}$ defines norm.

**Definition:** $(X, \langle \cdot, \cdot \rangle)$ is called a Hilbert space if $(X, \|\cdot\|_{\langle \cdot, \cdot \rangle})$ is a Banach space.

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# Functional analysis - part 9

## Examples of Hilbert spaces

(a) $\mathbb{R}^n$, $\mathbb{C}^n$ with $\langle x, y \rangle = \sum_{i=1}^{n} \overline{x_i}\, y_i$

(b) $\ell^2(\mathbb{N}, \mathbb{F})$ with $\langle x, y \rangle = \sum_{i=1}^{\infty} \overline{x_i}\, y_i$

Not a Hilbert space ⟶ (c) $C([0,1], \mathbb{F})$ with $\langle f, g \rangle = \int_0^1 \overline{f(t)}\, g(t)\, dt$ ↙ inner product

$(\ell^2(\mathbb{N}, \mathbb{F}), \langle \cdot, \cdot \rangle)$ is a Hilbert space: $\langle \cdot, \cdot \rangle : \ell^2 \times \ell^2 \longrightarrow \mathbb{F}$ ↗ later!

(1) positive definite: $\langle x, x \rangle = \sum_{i=1}^{\infty} \overline{x_i}\, x_i = \sum_{i=1}^{\infty} |x_i|^2 \geq 0$

and $\langle x, x \rangle = 0 \Rightarrow |x_i|^2 = 0$ for all $i \in \mathbb{N}$

$\Rightarrow x_i = 0$ for all $i \in \mathbb{N} \Rightarrow x = 0$.

(2) (conjugate) symmetric: $\overline{\langle y, x \rangle} = \sum_{i=1}^{\infty} \overline{\overline{y_i}\, x_i} = \sum_{i=1}^{\infty} y_i\, \overline{x_i} = \langle x, y \rangle$

(3) linear in the 2nd argument: $\langle x, y + z \rangle = \sum_{i=1}^{\infty} \overline{x_i}(y_i + z_i) = \sum_{i=1}^{\infty} \overline{x_i}\, y_i + \sum_{i=1}^{\infty} \overline{x_i}\, z_i$

$= \langle x, y \rangle + \langle x, z \rangle$

$\langle x, \lambda \cdot y \rangle = \sum_{i=1}^{\infty} \overline{x_i}(\lambda y_i) = \lambda \cdot \sum_{i=1}^{\infty} \overline{x_i}\, y_i = \lambda \cdot \langle x, y \rangle$

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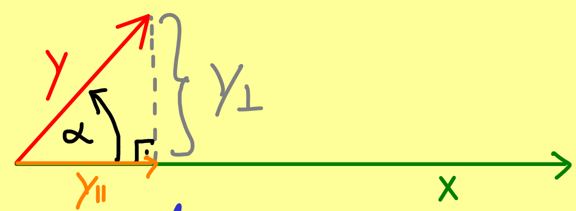
# Functional analysis - part 10

**Cauchy-Schwarz inequality**: Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $\|x\| := \sqrt{\langle x, x \rangle}$. Then for all $x, y \in X$:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$




and $|\langle x, y \rangle| = \|x\| \cdot \|y\| \iff x, y$ linearly dependent

**Proof:** 1st case: $x = 0$: $|\langle x, y \rangle| = 0 = \|x\| \cdot \|y\|$ ✓

2nd case $x \neq 0$: $\hat{x} := \frac{x}{\|x\|}$, $\quad y_\parallel := \langle \hat{x}, y \rangle \hat{x}$, $\quad y_\perp := y - y_\parallel$

$$0 \leq \|y_\perp\|^2 = \|y - y_\parallel\|^2 = \|y - \langle \hat{x}, y \rangle \hat{x}\|^2 = \langle y - \langle \hat{x}, y \rangle \hat{x}, y - \langle \hat{x}, y \rangle \hat{x} \rangle$$

$$= \langle y - \langle \hat{x}, y \rangle \hat{x}, y \rangle - \langle y - \langle \hat{x}, y \rangle \hat{x}, \langle \hat{x}, y \rangle \hat{x} \rangle$$

$$= \langle y, y \rangle - \langle \langle \hat{x}, y \rangle \hat{x}, y \rangle - \langle y, \langle \hat{x}, y \rangle \hat{x} \rangle + \langle \langle \hat{x}, y \rangle \hat{x}, \langle \hat{x}, y \rangle \hat{x} \rangle$$

$$= \|y\|^2 - \left( \langle \langle \hat{x}, y \rangle \hat{x}, y \rangle + \overline{\langle \langle \hat{x}, y \rangle \hat{x}, y \rangle} \right) + \|\langle \hat{x}, y \rangle \hat{x}\|^2$$

$$= \|y\|^2 - \left( 2 \cdot \mathrm{Re}\big( \underbrace{\langle \langle \hat{x}, y \rangle \hat{x}, y \rangle}_{= \overline{\langle \hat{x}, y \rangle} \langle \hat{x}, y \rangle = |\langle \hat{x}, y \rangle|^2} \big) \right) + |\langle \hat{x}, y \rangle|^2 \cdot \underbrace{\|\hat{x}\|^2}_{=1}$$

$$= \|y\|^2 - 2 |\langle \hat{x}, y \rangle|^2 + |\langle \hat{x}, y \rangle|^2 = \|y\|^2 - |\langle \hat{x}, y \rangle|^2$$

$$\Rightarrow \quad \|y\|^2 \geq |\langle \hat{x}, y \rangle|^2 = \left|\left\langle \frac{x}{\|x\|}, y \right\rangle\right|^2 = \frac{1}{\|x\|^2} |\langle x, y \rangle|^2$$

$$\Rightarrow \quad \|x\| \cdot \|y\| \geq |\langle x, y \rangle|$$

**$\Delta$-inequality for $\|\cdot\|$**:

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + 2\,\mathrm{Re}(\langle x, y \rangle) + \|y\|^2$$

$$\leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2$$

$$\overset{\text{Cauchy Schwarz}}{\leq} \|x\|^2 + 2 \cdot \|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

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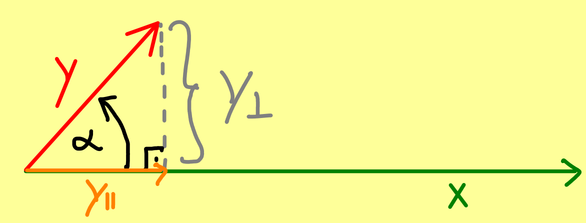
# Functional analysis - part 11




**Orthogonality:** Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space.

(a) $x, y \in X$ are called orthogonal if $\langle x, y\rangle = 0$. Write $x \perp y$.

(b) For $U, V \subseteq X$, write $U \perp V$ if $x \perp y$ for all $x \in U, y \in V$.

(c) For $U \subseteq X$, the orthogonal complement of $U$ is

$$\underbrace{U^\perp}_{\text{is always a subspace in } X} := \{ x \in X \mid \langle x, u\rangle = 0 \text{ for all } u \in U \}$$

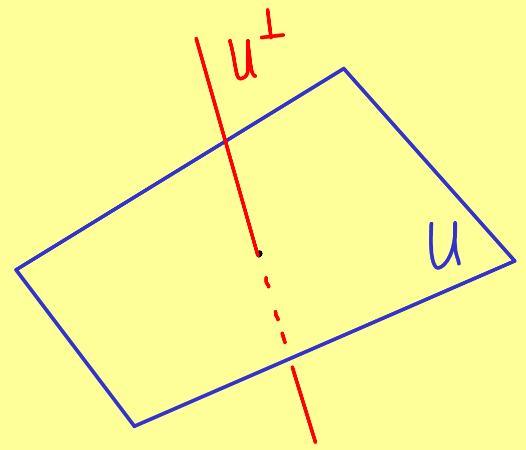
**Remark:** (1) $\{0\}^\perp = X$ , $X^\perp = \{0\}$

(2) $U \subseteq V \;\Rightarrow\; U^\perp \supseteq V^\perp$

Proof: $x \in V^\perp \Rightarrow \langle x, v\rangle = 0$ for all $v \in V$

$\overset{U \subseteq V}{\Rightarrow} \langle x, u\rangle = 0$ for all $u \in U \Rightarrow x \in U^\perp$

(3) If $x \perp y$, then $\|x+y\|^2_{\langle\cdot,\cdot\rangle} = \|x\|^2_{\langle\cdot,\cdot\rangle} + \|y\|^2_{\langle\cdot,\cdot\rangle}$ (Pythagorean theorem)




$U^\perp$ is always closed

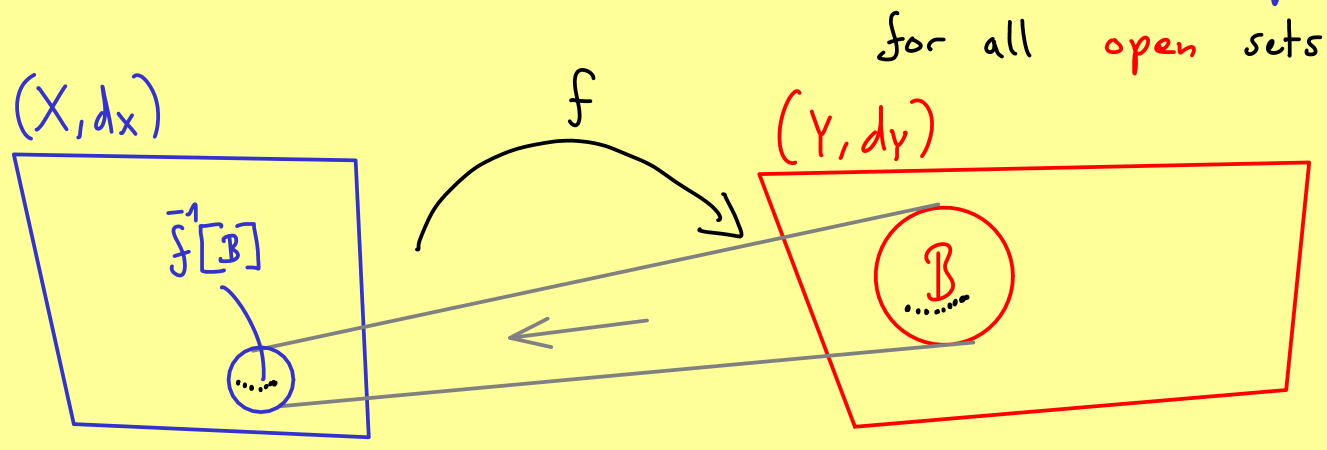
The Bright Side of Mathematics

# Functional analysis – part 12

**Continuity for metric spaces:** $(X, d_X)$, $(Y, d_Y)$ two metric spaces.

A map $f: X \to Y$ is called:

- **continuous** if $f^{-1}[B]$ is open (in $X$) for all open sets $B \subseteq Y$.




- **sequentially continuous** if for all $\tilde{x} \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n\to\infty} \tilde{x}$ holds $f(x_n) \xrightarrow{n\to\infty} f(\tilde{x})$.

**Fact:** For metric spaces, continuous and sequentially continuous are equivalent.

**Examples:** (a) $(X, d_X)$ discrete metric space, $(Y, d_Y)$ any metric space

$\Rightarrow$ all $f: X \to Y$ are continuous

(b) $(X, d_X)$, $(Y, d_Y)$ metric spaces, $y_0 \in Y$ fixed.

$\Rightarrow$ $f: X \to Y$, $x \mapsto y_0$ is always continuous.

(c) $(X, \|\cdot\|)$ normed space, $Y = \mathbb{R}$ with standard metric

$\Rightarrow$ $f: X \to \mathbb{R}$, $x \mapsto \|x\|$ is continuous

*Proof:* Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$f(x_n) = \|x_n\| = \|x_n - \tilde{x} + \tilde{x}\| \overset{\Delta\text{-inequ.}}{\leq} \|x_n - \tilde{x}\| + \|\tilde{x}\| = \underbrace{d(x_n, \tilde{x})}_{\xrightarrow{n\to\infty} 0} + f(\tilde{x})$$

$$\Rightarrow \lim_{n\to\infty} f(x_n) \leq f(\tilde{x})$$

$$f(\tilde{x}) = \|\tilde{x}\| = \|\tilde{x} - x_n + x_n\| \overset{\Delta\text{-inequ.}}{\leq} \|\tilde{x} - x_n\| + \|x_n\| = d(\tilde{x}, x_n) + f(x_n)$$

$$\Rightarrow f(\tilde{x}) \leq \lim_{n\to\infty} f(x_n) \qquad \square$$

(d) $(X, \langle\cdot,\cdot\rangle)$ inner product space, $Y = \mathbb{C}$ with the standard metric, $x_0 \in X$ fixed.

$\Rightarrow$ $f: X \to \mathbb{C}$, $x \mapsto \langle x_0, x\rangle$ is continuous

*Proof:* Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$|f(x_n) - f(\tilde{x})| = |\langle x_0, x_n\rangle - \langle x_0, \tilde{x}\rangle| = |\langle x_0, x_n - \tilde{x}\rangle|$$

$$\overset{\text{C.S.}}{\leq} \|x_0\| \cdot \underbrace{\|x_n - \tilde{x}\|}_{\xrightarrow{n\to\infty} 0} \xrightarrow{n\to\infty} 0$$

Analogously, $g: X \to \mathbb{C}$, $x \mapsto \langle x, x_0\rangle$ is continuous.

**Claim:** $(X, \langle\cdot,\cdot\rangle)$ inner product space, $U \subseteq X$. Then $U^\perp$ is closed.

*Proof:* Let $(x_n)_{n\in\mathbb{N}} \subseteq U^\perp$ with limit $\tilde{x} \in X$.

$\Rightarrow$ $\langle x_n, u\rangle = 0$ for all $u \in U$

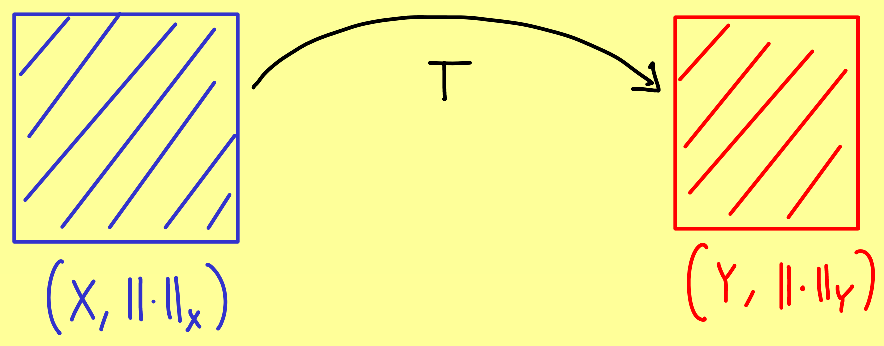
$\Rightarrow$ $\lim_{n\to\infty} \langle x_n, u\rangle = 0$ for all $u \in U$

$\Rightarrow$ $\langle \tilde{x}, u\rangle = 0$ for all $u \in U$ $\Rightarrow$ $\tilde{x} \in U^\perp$ $\qquad \square$

The Bright Side of Mathematics

# Functional analysis – part 13

Operator:

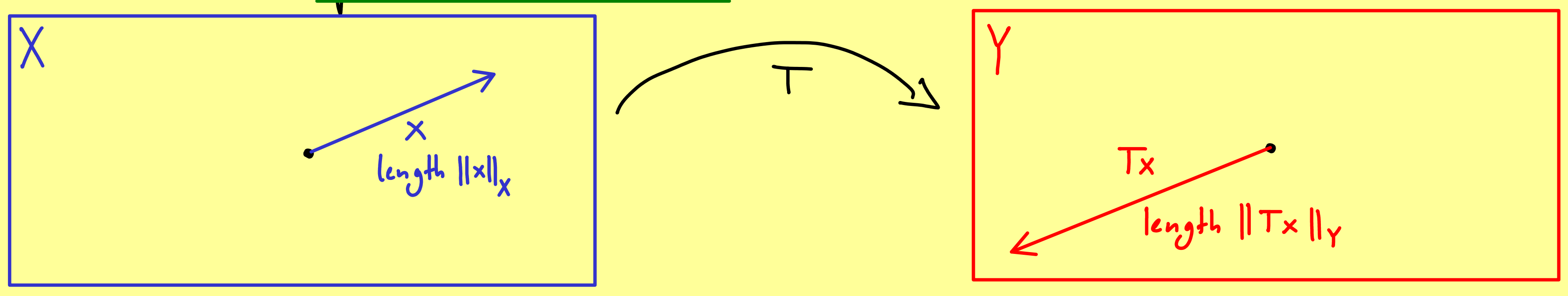



$T: X \longrightarrow Y$ :
- linear (conserves the algebraic structure)
- continuous (bounded) (conserves the topological structure)

**Definition:** $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed spaces, $T: X \longrightarrow Y$ linear $\left\{\begin{array}{l} T(x+\tilde{x}) = Tx + T\tilde{x} \\ T(\lambda x) = \lambda Tx \end{array}\right.$ for all $x, \tilde{x} \in X$, $\lambda \in \mathbb{F}$

$$\|T\| = \|T\|_{X \to Y} := \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \;\middle|\; x \in X,\ x \neq 0 \right\}$$

is called the operator norm of $T$. If $\|T\| < \infty$, $T$ is called bounded.




**Proposition:** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed spaces, $T: X \longrightarrow Y$ linear. Then the following claims are equivalent:

(a) $T$ is continuous.
(b) $T$ is continuous at $x = 0$.
(c) $T$ is bounded.

**Proof:** (a) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (c): (*) For all sequences $(x_n)_{n \in \mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n \to \infty} 0$, we have $Tx_n \xrightarrow{n \to \infty} 0$.

Claim: (*) $\Rightarrow$ [There is a $\delta > 0$ such that $\|Tx\|_Y < 1$ for all $x \in X$ with $\|x\|_X < \delta$] (*)

Proof of the claim: $\neg$(*) $\Rightarrow$ For all $n \in \mathbb{N}$, we find $x_n \in X$ with $\|x_n\|_X < \frac{1}{n}$ and $\|Tx_n\|_Y \geq 1$ $\Rightarrow$ $\neg$(*)

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|Tx\|_Y \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}}{\|x\|_X \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}} = \frac{\left\| T\left(\frac{\delta}{2} \frac{x}{\|x\|_X}\right) \right\|_Y}{\underbrace{\left\| \frac{\delta}{2} \frac{x}{\|x\|_X} \right\|_X}_{= \frac{\delta}{2}}} \leq \frac{2}{\delta}$$

$$\Rightarrow \|T\| = \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \;\middle|\; x \in X,\ x \neq 0 \right\} \leq \frac{2}{\delta} < \infty$$

(c) $\Rightarrow$ (a): Let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be convergent with limit $\tilde{x} \in X$. Then

$$\|Tx_n - T\tilde{x}\|_Y = \|T(x_n - \tilde{x})\|_Y \leq \|T\| \cdot \|x_n - \tilde{x}\|_X \xrightarrow{n \to \infty} 0 \qquad \square$$

# Functional analysis – part 14

**Example:** $X = (C([0,1], \mathbb{F}), \|\cdot\|_\infty)$ , $Y = (\mathbb{F}, |\cdot|)$

For $g \in X$ with $g(t) \neq 0$ for all $t \in [0,1]$, define

$$T_g : X \longrightarrow Y \quad \text{by} \quad T_g(f) := \int_0^1 g(t) \cdot f(t)\, dt$$

What is $\|T_g\|$?

$$\|T_g\| = \sup\left\{ \frac{|T_g(f)|}{\|f\|_\infty} \;\middle|\; f \in X,\, f \neq 0 \right\}$$

$$= \sup\left\{ |T_g(f)| \;\middle|\; f \in X,\, \|f\|_\infty = 1 \right\}$$

$$= \sup\left\{ \underbrace{\left| \int_0^1 g(t) \cdot f(t)\, dt \right|}_{\leq \int_0^1 |g(t)| \cdot \underbrace{|f(t)|}_{\leq \|f\|_\infty = 1} dt} \;\middle|\; f \in X,\, \|f\|_\infty = 1 \right\}$$

$$\leq \int_0^1 |g(t)|\, dt < \infty$$

Check the other inequality: $h(t) := \dfrac{\overline{g(t)}}{|g(t)|}$ with $\|h\|_\infty = 1$

$$\|T_g\| \geq |T_g(h)| = \left| \int_0^1 g(t) \frac{\overline{g(t)}}{|g(t)|}\, dt \right| = \int_0^1 \frac{|g(t)|^{\cancel{2}}}{\cancel{|g(t)|}}\, dt = \int_0^1 |g(t)|\, dt$$

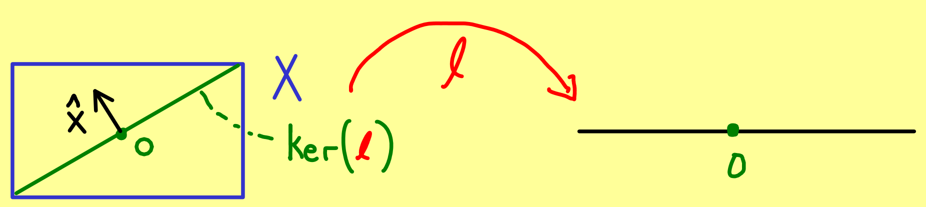
# Functional analysis – part 15

## Riesz representation theorem

Let $(X,\langle\cdot,\cdot\rangle)$ be a Hilbert space. Then for each continuous linear map $\ell: X \to \mathbb{F}$ (a continuous linear functional) there is exactly one $x_\ell \in X$ such that $\ell(x) = \langle x_\ell, x\rangle$ for all $x\in X$ and $\|\ell\|_{X\to\mathbb{F}} = \|x_\ell\|_X$.

[In physics $\ell = \langle \psi |$ ]

Proof: (1) Existence:




First case: $\ker(\ell) = X \Rightarrow x_\ell = 0$

Second case: $\ker(\ell) \neq X \rightsquigarrow x_\ell \in \underbrace{\ker(\ell)^\perp}_{\supsetneq \{0\}}$ true because $\ker(\ell)$ is closed and "orthogonal projections" exist in Hilbert spaces (later)

Kernel is preimage of closed set $\{0\}$ $\overset{\text{continuity}}{\rightsquigarrow}$ Kernel is closed.

Choose $\hat{x} \in \ker(\ell)^\perp$ with $\|\hat{x}\|_X = 1$. Set $x_\ell := \overline{\ell(\hat{x})}\cdot \hat{x}$

$$\ell(x) = \ell\left(x - \frac{\ell(x)}{\ell(\hat{x})}\hat{x} + \frac{\ell(x)}{\ell(\hat{x})}\hat{x}\right) = \underbrace{\ell\left(x - \frac{\ell(x)}{\ell(\hat{x})}\hat{x}\right)}_{\ell(x) - \frac{\ell(x)}{\cancel{\ell(\hat{x})}}\cdot\cancel{\ell(\hat{x})} = 0} + \ell\Big(\underbrace{\frac{\ell(x)}{\ell(\hat{x})}}_{\lambda}\hat{x}\Big)$$

$$= \lambda\cdot\ell(\hat{x})\cdot\langle\hat{x},\hat{x}\rangle = \lambda\cdot\langle\overline{\ell(\hat{x})}\hat{x}, \hat{x}\rangle = \langle x_\ell, \lambda\hat{x}\rangle$$

$$= \langle x_\ell, \underbrace{\lambda\hat{x} - x}_{\in\ker(\ell)} + x\rangle = \langle x_\ell, x\rangle$$

(2) Uniqueness: Assume $x_\ell, \tilde{x}_\ell \in X$ fulfil $\ell(x) = \langle x_\ell, x\rangle = \langle \tilde{x}_\ell, x\rangle$

$$\Rightarrow \langle x_\ell - \tilde{x}_\ell, x\rangle = 0 \quad \text{for all } x\in X.$$

$$\Rightarrow \langle x_\ell - \tilde{x}_\ell, x_\ell - \tilde{x}_\ell\rangle = 0 \quad \Rightarrow \quad x_\ell = \tilde{x}_\ell$$

(3) Operator norm:

$$\|\ell\| = \sup\left\{ |\ell(x)| \;\middle|\; \|x\|_X = 1\right\} = \sup\left\{ \underbrace{|\langle x_\ell, x\rangle|}_{\leq \|x_\ell\|_X\cdot\|x\|_X} \;\middle|\; \|x\|_X = 1\right\}$$

$$\leq \|x_\ell\|$$

$$\|\ell\| \geq \left|\ell\left(\frac{x_\ell}{\|x_\ell\|}\right)\right| = \left|\left\langle x_\ell, \frac{x_\ell}{\|x_\ell\|}\right\rangle\right| = \|x_\ell\|$$

$\square$

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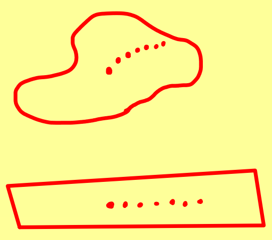
# Functional analysis – part 16

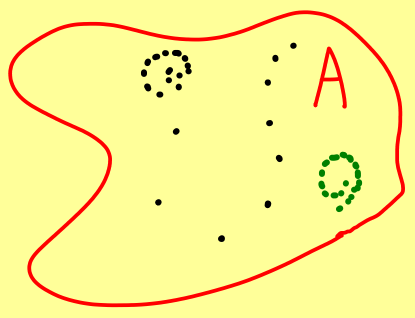
## Compactness $\mathbb{R}^n \supseteq A$

A is compact $\overset{\text{only in } \mathbb{R}^n \text{ or } \mathbb{C}^n}{=}$ $\begin{cases} \bullet\ A \text{ is closed} \\ \bullet\ A \text{ is bounded} \end{cases}$



**Definition:** Let $(X,d)$ be a metric space. $A \subseteq X$ is called (sequentially) compact if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ one finds a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with

$$\tilde{x} := \lim_{k\to\infty} x_{n_k} \in A$$




**Examples:** (a) $(\mathbb{R}, d_{eucl.})$, $A = [0,1]$ compact by Bolzano-Weierstrass theorem.

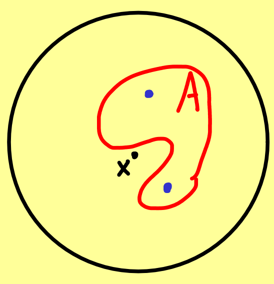
(b) $(\mathbb{R}, d_{discr.})$, $A = [0,1]$ not compact because:

The sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ with $x_n = \frac{1}{n}$ satisfies

$d_{discr.}(x_n, x_m) = 1$ for all $n,m \in \mathbb{N}$ with $n \neq m$.

$\Rightarrow$ no convergent subsequence

**Proposition:** Let $(X,d)$ be a metric space and $A \subseteq X$ compact. Then $A$ is closed and bounded. There is an $x \in X$ and an $\varepsilon > 0$ such that $B_\varepsilon(x) \supseteq A$



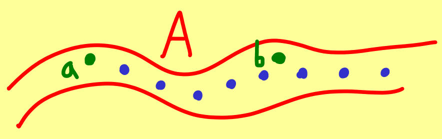

**Proof:** Let $A \subseteq X$ be compact.

(1) Let $(x_n)_{n\in\mathbb{N}} \subseteq A$ be convergent with limit $\tilde{x} \in X$.

$\overset{\text{compact}}{\Longrightarrow}$ There is a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit $\tilde{\tilde{x}} \in A$

$\overset{\text{limit unique}}{\Longrightarrow}$ $\tilde{x} = \tilde{\tilde{x}} \in A \quad \Rightarrow \quad A$ is closed

(2) Contraposition: $A$ is not bounded




$\Rightarrow$ For given $a \in A$, there are $x_n \in A$ with $d(a, x_n) > n$.

$\Rightarrow$ For any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and any point $b \in A$:

$$n_k < d(a, x_{n_k}) \leq d(a,b) + d(b, x_{n_k})$$

$$\Rightarrow n_k - d(a,b) \leq d(b, x_{n_k})$$

$\Rightarrow d(b, x_{n_k}) \overset{k\to\infty}{\not\longrightarrow} 0$ for all $b \in A \quad \Rightarrow A$ not compact

The Bright Side of Mathematics

# Functional analysis – part 17

## Arzelà-Ascoli theorem

Example: (a) $(X, \|\cdot\|)$ normed space with $\dim(X) < \infty$ (always Banach space)

$A \subseteq X$: $A$ compact $\Longleftrightarrow$ $A$ closed + bounded

(b) $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ for $p \in [1,\infty)$ (Banach space)

$A := \{ x \in \ell^p(\mathbb{N}) \mid \|x\|_p \leq 1 \}$ closed + bounded

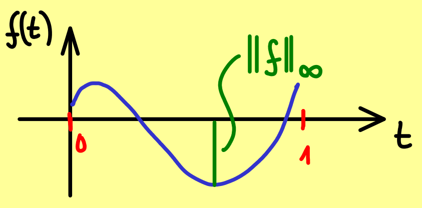
$$\left.\begin{array}{l} e_1 := (1,0,0,0,\dots) \in A \\ e_2 := (0,1,0,0,\dots) \in A \\ e_3 := (0,0,1,0,\dots) \in A \\ \vdots \end{array}\right\} \quad (e_n)_{n\in\mathbb{N}} \subseteq A$$

$$\|e_n - e_m\|_p \overset{n \neq m}{=} \sqrt[p]{|1|^p + |-1|^p} = \sqrt[p]{2}$$

$\Rightarrow$ no convergent subsequence

## Continuous functions: $(C([0,1]), \|\cdot\|_\infty)$, $\|f\|_\infty := \sup\{ |f(t)| \mid t \in [0,1] \}$

$\hookrightarrow$ Banach space




$f$ is called uniformly continuous: (Using $\varepsilon$-$\delta$-characterisation)

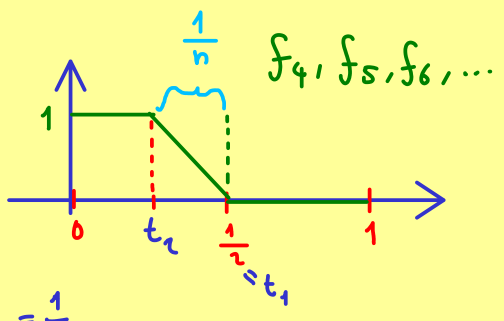
$$\forall_{\varepsilon>0} \; \exists_{\delta>0} \; \forall_{t_1,t_2 \in [0,1]} : \; |t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon$$

$A \subseteq C([0,1])$ is called uniformly equicontinuous:

$$\forall_{\varepsilon>0} \; \exists_{\delta>0} \; \forall_{t_1,t_2 \in [0,1]} \; \forall_{f \in A} : \; |t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon$$

or equivalently $\quad \sup_{f \in A} |f(t_1) - f(t_2)| \xrightarrow{|t_1 - t_2| \to 0} 0$

Examples: (a) $A := \{ f \in C([0,1]) \mid \|f\|_\infty \leq 1 \}$




$$\sup_{f \in A} |f(t_1) - f(t_2)| \geq |f_n(t_1) - f_n(t_2)| \quad \text{for } t_1 = \tfrac{1}{2}, \; t_2 = \tfrac{1}{2} - \tfrac{1}{n}$$

$= 1$ (for $n \geq 4$)

$\Rightarrow$ $A$ is not equicontinuous!

(b) $A := \{ f \in C([0,1]) \mid f \text{ continuously differentiable}, \; \|f'\|_\infty \leq 2 \}$

$$|f(t_1) - f(t_2)| \overset{\text{mean value theorem}}{\leq} |f'(\xi)| \cdot |t_1 - t_2| \leq 2 \cdot |t_1 - t_2|$$

$$\sup_{f \in A} |f(t_1) - f(t_2)| \leq 2 \cdot |t_1 - t_2| \xrightarrow{|t_1 - t_2| \to 0} 0 \quad \Rightarrow \quad A \text{ is uniformly equicontinuous}$$

**Arzelà-Ascoli theorem:** For $(C([0,1]), \|\cdot\|_\infty)$ holds: ($[0,1]$: could be any compact metric space)

$$A \subseteq C([0,1]) \text{ compact} \quad \Longleftrightarrow \quad A \text{ is } \left\{ \begin{array}{l} \text{closed +} \\ \text{bounded +} \\ \text{uniformly equicontinuous} \end{array} \right.$$

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## Functional analysis – part 18

**Compact operators:** $T: \mathbb{F}^n \longrightarrow \mathbb{F}^m$ linear (standard norm)

$\Rightarrow$ $T$ is continuous / bounded

$\Rightarrow$ $T[B_1(0)] \subseteq \mathbb{F}^m$ bounded

$\Rightarrow$ $\overline{T[B_1(0)]} \subseteq \mathbb{F}^m$ compact

However: $I: \ell^p(\mathbb{N}) \longrightarrow \ell^p(\mathbb{N})$, $p \in [1,\infty)$,

$x \longmapsto x$ $\Rightarrow$ $\overline{I[B_1(0)]} = \overline{B_1(0)}$ not compact (closed unit ball in $\ell^p(\mathbb{N})$)

**Definition:** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces. A bounded linear operator $T: X \to Y$ is called compact if $\overline{T[B_1(0)]} \subseteq Y$ is a compact set.

**Example:** Integral operator $T_k: C([0,1]) \longrightarrow C([0,1])$ for $k \in C([0,1]\times[0,1])$ (with supremum norm $\|\cdot\|_\infty$)

$$(T_k f)(s) := \int_0^1 k(s,t)\, f(t)\, dt$$

**Fact:** $k$ is uniformly continuous:

$$\forall_{\varepsilon>0} \; \exists_{\delta>0} \; \forall_{(s_1,t_1),(s_2,t_2)} \quad \|(s_1,t_1)-(s_2,t_2)\| < \delta \Rightarrow |k(s_1,t_1) - k(s_2,t_2)| < \varepsilon$$

For $\varepsilon > 0$, choose $\delta > 0$ such that . Therefore for $s_1, s_2 \in [0,1]$ with $|s_1 - s_2| < \delta$:

$$|(T_k f)(s_1) - (T_k f)(s_2)| = \left| \int_0^1 \big(k(s_1,t)\, f(t) - k(s_2,t)\, f(t)\big) dt \right|$$

$$\leq \int_0^1 \underbrace{|k(s_1,t) - k(s_2,t)|}_{<\varepsilon} \cdot \underbrace{|f(t)|}_{\leq \|f\|_\infty} \, dt < \varepsilon \cdot \|f\|_\infty$$

$A := T_k[B_1(0)]$. We have:

$$\forall_{\varepsilon>0} \; \exists_{\delta>0} \; \forall_{s_1,s_2\in[0,1]} \; \forall_{g \in A} : |s_1 - s_2| < \delta \Rightarrow |g(s_1) - g(s_2)| < \varepsilon$$

$\Rightarrow$ $T_k[B_1(0)]$ is uniformly equicontinuous

**Boundedness:**
$$\|T_k\| = \sup\left\{ \|T_k f\|_\infty \;\middle|\; \|f\|_\infty = 1 \right\}$$
$$= \sup\left\{ \sup_{s\in[0,1]} \left| \int_0^1 k(s,t) f(t)\, dt \right| \;\middle|\; \|f\|_\infty = 1 \right\}$$
$$\leq \sup\left\{ \sup_{s\in[0,1]} \int_0^1 |k(s,t)|\, \underbrace{|f(t)|}_{\leq\|f\|_\infty}\, dt \;\middle|\; \|f\|_\infty = 1 \right\}$$
$$\leq \sup_{s\in[0,1]} \int_0^1 |k(s,t)|\, dt \leq \|k\|_\infty$$

$\Rightarrow$ By Arzelà-Ascoli: $\overline{T_k[B_1(0)]}$ is compact $\Rightarrow$ $T_k$ compact operator

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# Functional analysis – part 19

## Hölder's inequality (for $\mathbb{F}^n$ and $p \in (1,\infty)$)

$\hookrightarrow p' \in (1, \infty)$ Hölder conjugate

For $x \in \mathbb{F}^n$:

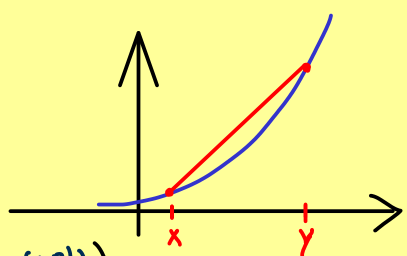
$$\|x\|_q := \left( \sum_{j=1}^{n} |x_j|^q \right)^{\frac{1}{q}} , \quad q \in [1,\infty) \qquad \boxed{\frac{1}{p} + \frac{1}{p'} = 1}$$

For $x, y \in \mathbb{F}^n$ write: $\quad xy := \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_n y_n \end{pmatrix}$

Then: $\|xy\|_1 \leq \|x\|_p \cdot \|y\|_{p'}$ for all $x, y \in \mathbb{F}^n$

## Young's inequality: $a, b > 0 \implies a \cdot b \leq \dfrac{a^p}{p} + \dfrac{b^{p'}}{p'}$

Proof: $f : x \mapsto e^x$ is convex: $\lambda \in [0,1]$




$$\underbrace{a \cdot b}_{} = f(\log(a) + \log(b)) = f(\underbrace{\lambda}_{\frac{1}{p}} \underbrace{x}_{\log(a^p)} + \underbrace{(1-\lambda)}_{\frac{1}{p'}} \underbrace{y}_{\log(b^{p'})}) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$= \frac{1}{p} f(\log(a^p)) + \frac{1}{p'} f(\log(b^{p'}))$$

$$= \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$$

## Proof of Hölder's inequality: $1^{st}$ case: $x = 0$ or $y = 0$

$2^{nd}$ case:

$$\frac{1}{\|x\|_p \cdot \|y\|_{p'}} \|xy\|_1 = \frac{1}{\|x\|_p \cdot \|y\|_{p'}} \sum_{j=1}^{n} |x_j y_j| = \sum_{j=1}^{n} \overbrace{\frac{|x_j|}{\|x\|_p}}^{a} \cdot \overbrace{\frac{|y_j|}{\|y\|_{p'}}}^{b}$$

$$\leq \sum_{j=1}^{n} \frac{1}{p} \cdot \frac{|x_j|^p}{\|x\|_p^p} + \sum_{j=1}^{n} \frac{1}{p'} \frac{|y_j|^{p'}}{\|y\|_{p'}^{p'}} = \frac{1}{p} + \frac{1}{p'} = 1$$

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# Functional analysis – part 20

Minkowski's inequality: $\Delta$-inequality for $\|\cdot\|_p$ in $\ell^p(\mathbb{N})$:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \text{for all } x,y \in \ell^p(\mathbb{N}), \quad p \in [1,\infty)$$

Proof: For $p=1$: $\|x+y\|_1 = \sum_{j=1}^{\infty} \underbrace{|x_j+y_j|}_{\leq |x_j|+|y_j|} \leq \|x\|_1 + \|y\|_1$

For $p \in (1,\infty)$: Hölder conjugate $p' \in (1,\infty)$ $\quad \boxed{\frac{1}{p} + \frac{1}{p'} = 1}$

$$\frac{p}{p-1} = p'$$

$$\|x+y\|_p^p = \sum_{j=1}^{\infty} |x_j+y_j|^p = \lim_{n\to\infty} \sum_{j=1}^{n} \underbrace{|x_j+y_j|^p}_{\leq |x_j|+|y_j|} = (*)$$

$$(**) \quad (|x_j|+|y_j|)^p = (|x_j|+|y_j|)(|x_j|+|y_j|)^{p-1} = \underbrace{|x_j|}_{a_j}\underbrace{(|x_j|+|y_j|)^{p-1}}_{b_j} + \underbrace{|y_j|}_{c_j}(|x_j|+|y_j|)^{p-1} \rightsquigarrow a,b,c \in \mathbb{F}^n$$

Hölder: $\|ab\|_1 \leq \|a\|_p \cdot \underbrace{\|b\|_{p'}}_{}$

$$= \left(\sum_{j=1}^{n} \left|(|x_j|+|y_j|)^{p-1}\right|^{p'}\right)^{\frac{1}{p'}} = \left(\sum_{j=1}^{n} (|x_j|+|y_j|)^p\right)^{\frac{1}{p'}}$$

$$(***) \quad \sum_{j=1}^{n} (|x_j|+|y_j|)^p \leq \|a\|_p \cdot \|b\|_{p'} + \|c\|_p \cdot \|b\|_{p'} = (\|a\|_p + \|c\|_p) \cdot \left(\sum_{j=1}^{n} (|x_j|+|y_j|)^p\right)^{\frac{1}{p'}}$$

$$\Rightarrow \left(\sum_{j=1}^{n} (|x_j|+|y_j|)^p\right)^{\overbrace{\frac{1}{p}}^{1-\frac{1}{p'}}} \leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}$$

$$\xrightarrow[+ (*)]{n\to\infty} \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

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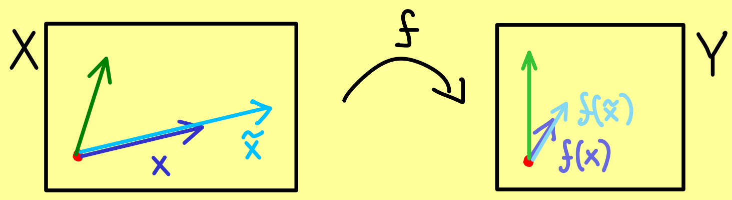
# Functional analysis – part 21

## Isomorphisms?

Homomorphism : map that preserves structures

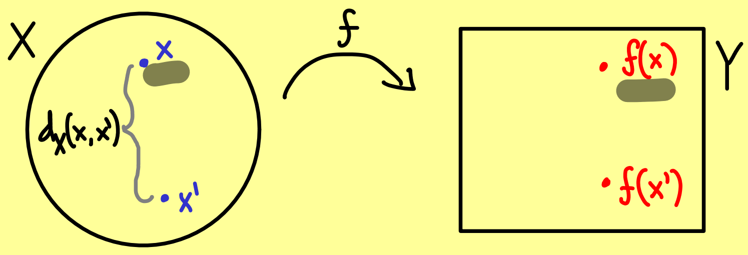
Example: (a) Let $X, Y$ be vector spaces and $f: X \to Y$ be a map.




$$\left.\begin{aligned} f(\lambda \cdot x) &= \lambda \cdot f(x) \\ f(x + x') &= f(x) + f(x') \end{aligned}\right\} \text{linear}$$

homomorphism = linear map

(b) Let $(X, d_X)$, $(Y, d_Y)$ be two metric spaces and $f: X \to Y$ be a map.




$$d_Y(f(x), f(x')) \leq d_X(x, x') \quad (*)$$

homomorphism = map that satisfies $(*)$

$$\boxed{\text{isomorphism} = \text{homomorphism} + \text{bijective} + \text{inverse map is also homomorphism}}$$

Isomorphism for Banach spaces $X, Y$:

$f: X \to Y$ with: linear + bijective + $\|f(x)\|_Y = \|x\|_X$

(often called isometric isomorphism)

Example: (a) $S_R : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$, $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$

$\Rightarrow$ linear, $\|S_R x\|_p = \|x\|_p$ not surjective $\Rightarrow$ not an isomorphism

(b) $S: \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$, $(\dots, x_{-1}, x_0, x_1, x_2, \dots) \mapsto (\dots, x_{-2}, x_{-1}, x_0, x_1, \dots)$

index … -1 0 1 2 …     index … -1 0 1 2 …

$\Rightarrow$ linear, $\|S x\|_p = \|x\|_p$ and bijective $\Rightarrow$ isomorphism

# Functional analysis – part 22

**Dual spaces:** $X$ normed space $\rightsquigarrow$ $X'$ normed space

$$X' := \{ \ell : X \to \mathbb{F} \mid \ell \text{ linear + bounded} \}$$

**Recall the Riesz representation theorem:** $X$ Hilbert space. Then: $X' \xleftrightarrow{\text{isometric isomorphism}} X$

**Proposition:** Let $X$ be a normed space. Then $(X', \|\cdot\|_{X\to\mathbb{F}})$ is a Banach space.

**Proof:** Let $(\ell_k)_{k\in\mathbb{N}} \subseteq X'$ be a Cauchy sequence:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n,m \geq N : \qquad \|\ell_n - \ell_m\|_{X\to\mathbb{F}} < \varepsilon$$

$$\|\ell_n - \ell_m\|_{X\to\mathbb{F}} \geq \frac{1}{\|x\|_X} |\ell_n(x) - \ell_m(x)| \quad \text{for } x \in X,\ x \neq 0.$$

$\Rightarrow (\ell_k(x))_{k\in\mathbb{N}} \subseteq \mathbb{F}$ is Cauchy sequence for all $x \in X$.

$$\Rightarrow \quad \ell(x) := \lim_{k\to\infty} \ell_k(x), \quad \ell : X \to \mathbb{F}$$

Show:
(1) $\ell$ is linear ✓
(2) $\ell$ is bounded ✓
(3) $\|\ell_k - \ell\|_{X\to\mathbb{F}} \xrightarrow{k\to\infty} 0$ ✓

**For (2):** $\|\ell_n\|_{X\to\mathbb{F}} \leq \underbrace{\|\ell_n - \ell_N\|_{X\to\mathbb{F}}}_{<\varepsilon} + \underbrace{\|\ell_N\|_{X\to\mathbb{F}}}_{=:C} \leq C + \varepsilon$ for all $n \geq N$

$$\Rightarrow \quad |\ell(x)| = \left|\lim_{k\to\infty} \ell_k(x)\right| = \lim_{k\to\infty} |\ell_k(x)| \leq \lim_{k\to\infty} \underbrace{\|\ell_k\|_{X\to\mathbb{F}}}_{\leq \tilde{C}} \|x\|_X$$

$$\Rightarrow \quad \|\ell\|_{X\to\mathbb{F}} \leq \tilde{C} < \infty$$

**For (3):** For $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that for all $n,m \geq N$:

$$\frac{1}{\|x\|_X} |\ell_n(x) - \ell_m(x)| < \varepsilon$$

$$\Rightarrow \quad \sup_{\substack{x\in X \\ x\neq 0}} \frac{1}{\|x\|_X} \Big|\ell_n(x) - \overbrace{\lim_{m\to\infty} \ell_m(x)}^{\ell(x)}\Big| \leq \varepsilon \quad \Rightarrow \quad \|\ell_n - \ell\|_{X\to\mathbb{F}} \leq \varepsilon$$

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# Functional analysis – part 23

**Dual space:** $X$ normed space

$X' := \{ \ell : X \to \mathbb{F} \mid \ell \text{ linear + bounded} \}$

**Example:** $X = \ell^p(\mathbb{N})$ for $p \in (1, \infty)$

$X' \cong \ell^{p'}(\mathbb{N})$ where $p' \in (1,\infty)$ Hölder conjugate $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$

there is an isometric isomorphism

$T : \ell^{p'}(\mathbb{N}) \longrightarrow (\ell^p(\mathbb{N}))'$

$(Tx)(y) := \sum_{k=1}^{\infty} x_k \cdot y_k$ or $x \mapsto \langle \overline{x}, \cdot \rangle_{\ell^2(\mathbb{N})}$

**To show:**
(1) $T$ is well-defined ✓
(2) $T$ is linear ✓
(3) $T$ is bounded ✓
(4) $T$ surjective
(5) $\|Tx\| = \|x\|$ for all $x \in \ell^{p'}(\mathbb{N})$ (isometric)

**Proof:** (1) $|(Tx)(y)| \leq \lim_{n\to\infty} \sum_{k=1}^{n} |y_k \cdot x_k| \overset{\text{Hölder}}{\leq} \|y\|_p \cdot \|x\|_{p'} < \infty$

$\Rightarrow Tx$ is linear and bounded for all $x \in \ell^{p'}(\mathbb{N})$

(2) $T$ is linear.

(3) $\|Tx\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} = \sup \{ \underbrace{|(Tx)(y)|}_{\leq \|y\|_p \cdot \|x\|_{p'}} \mid \|y\|_p = 1 \} \leq \|x\|_{p'}$

$\boxed{T : \ell^{p'}(\mathbb{N}) \longrightarrow (\ell^p(\mathbb{N}))'}$

$\Rightarrow \|T\| \leq 1$

(4) Let $y' \in (\ell^p(\mathbb{N}))'$ and $e_k = (0, 0, \ldots, 0, \overset{k^{\text{th}} \text{ position}}{1}, 0, \ldots)$.

Define: $x_k := y'(e_k)$ and $x := (x_k)_{k\in\mathbb{N}}$

Question: $x \in \ell^{p'}(\mathbb{N})$ and $Tx = y'$ ?

$$\sum_{k=1}^{n} |x_k|^{p'} = \sum_{k=1}^{n} x_k \cdot t_k \qquad t_k = \begin{cases} \frac{|x_k|^{p'}}{x_k}, & x_k \neq 0 \\ 0, & x_k = 0 \end{cases}$$

$$= \sum_{k=1}^{n} t_k \cdot y'(e_k) = y'\left(\sum_{k=1}^{n} t_k e_k\right)$$

$$\leq \|y'\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} \cdot \underbrace{\Big\| \overbrace{\sum_{k=1}^{n} t_k e_k}^{= (t_1, t_2, \ldots, t_n, 0, 0, \ldots)} \Big\|_p}_{\left(\sum_{k=1}^{n} |t_k|^p\right)^{\frac{1}{p}}}$$

$$|t_k|^p = \left(\frac{|x_k|^{p'}}{|x_k|}\right)^p = |x_k|^{(p'-1)\cdot p} = |x_k|^{p'}$$

$\boxed{\left(\frac{1}{p} + \frac{1}{p'} = 1\right)}$

$$= \|y'\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} \left(\sum_{k=1}^{n} |x_k|^{p'}\right)^{\frac{1}{p}}$$

$\overset{n\to\infty}{\Longrightarrow} \|x\|_{p'} \leq \|y'\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} \quad \Longrightarrow \quad x \in \ell^{p'}(\mathbb{N})$ ✓

For $y \in \ell^p(\mathbb{N})$: $(Tx - y')(y) = (Tx - y')\left(\lim_{n\to\infty} \sum_{k=1}^{n} y_k e_k\right)$

$\boxed{(Tx)(y) := \sum_{j=1}^{\infty} x_j \cdot y_j}$

$$\overset{\text{continuity}}{=} \lim_{n\to\infty} (Tx - y')\left(\sum_{k=1}^{n} y_k e_k\right)$$

$$\overset{\text{linearity}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} y_k \cdot (Tx - y')(e_k) = 0 \qquad \text{surjective ✓}$$

(5) $\|Tx\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} \underset{=}{\leq} \|x\|_{p'} \underset{=}{\leq} \|y'\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} = \|Tx\|_{\ell^p(\mathbb{N}) \to \mathbb{F}}$ isometry ✓

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## Functional analysis - part 24

### Uniform boundedness principle (Banach-Steinhaus theorem)

$X, Y$ normed spaces, $X$ Banach space.

$$\mathcal{B}(X,Y) := \{ T : X \to Y \mid T \text{ linear + bounded} \}$$

**Theorem:** For every subset $\mathcal{M} \subseteq \mathcal{B}(X,Y)$ holds:

$\mathcal{M}$ is bounded pointwise on $X$ $\Longleftrightarrow$ $\mathcal{M}$ is uniformely bounded

More concretely: $\underset{x \in X}{\forall} \; \underset{C_x \geq 0}{\exists} \; \underset{T \in \mathcal{M}}{\forall} \; \|Tx\|_Y \leq C_x \Longleftrightarrow \underset{C \geq 0}{\exists} \; \underset{T \in \mathcal{M}}{\forall} \; \|T\|_{X \to Y} \leq C$

**Proposition:** $X, Y$ normed spaces, $X$ Banach space.

Let $T_n \in \mathcal{B}(X,Y)$ for all $n \in \mathbb{N}$ with $\lim\limits_{n \to \infty} T_n x$ exists for all $x \in X$.

Then: $T : X \to Y$ defined by $Tx := \lim\limits_{n \to \infty} T_n x$ is linear and bounded.

**Proof:** $\mathcal{M} := \{ T_n \mid n \in \mathbb{N} \}$ is bounded pointwise on $X$ $\overset{\text{Banach-Steinhaus}}{\Longrightarrow}$ There is a $C \geq 0$ with $\|T_n\| \leq C$ for all $n$

$$\Rightarrow \|T\|_{X \to Y} = \sup \{ \|Tx\|_Y \mid \|x\|_X = 1 \} \leq C$$

$$\|\lim_{n \to \infty} T_n x\|_Y = \lim_{n \to \infty} \|T_n x\|_Y \leq C$$

# Functional analysis – part 25

**Hahn-Banach theorem** $(X, \|\cdot\|_X)$ normed space $\rightsquigarrow (X', \|\cdot\|_{X'})$

$U \subseteq X$ subspace, $u' : U \longrightarrow \mathbb{F}$ continuous linear functional

Then: There exists $x' : X \longrightarrow \mathbb{F}$ continuous linear functional

with $x'(u) = u'(u)$ for all $u \in U$,

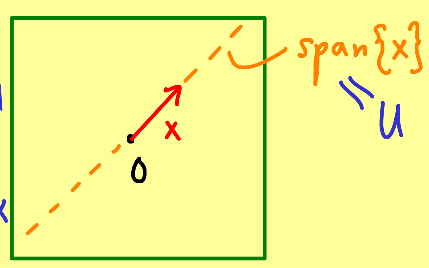
$\|x'\|_{X'} = \|u'\|_{U'}$ .

**Applications**: $(X, \|\cdot\|_X)$ normed space

(a) For all $x \in X$, $x \neq 0$, there is an $x' \in X'$ with $\|x'\|_{X'} = 1$ and $x'(x) = \|x\|_X$ .

**Proof:** Define $u' : U \longrightarrow \mathbb{F}$ continuous linear functional

$\lambda \cdot x \longmapsto \lambda \cdot \|x\|_X$




$\overset{\text{Hahn-Banach}}{\Longrightarrow} \; x' : X \longrightarrow \mathbb{F}$ with $x'(x) = u'(x) = \|x\|_X$

$\|x'\|_{X'} = \|u'\|_{U'} = 1$

(b) $X'$ separates the points of $X$: For $x_1, x_2 \in X$, $x_1 \neq x_2$,

there is an $x' \in X'$ with $x'(x_1) \neq x'(x_2)$

**Proof:** $x := x_2 - x_1 \overset{(a)}{\Longrightarrow} \underbrace{x'(x)}_{= x'(x_2) - x'(x_1)} = \|x\|_X \neq 0 \Rightarrow x'(x_1) \neq x'(x_2)$

(c) For all $x \in X$: $\|x\|_X = \sup\{ |x'(x)| \mid x' \in X', \|x'\| = 1 \}$

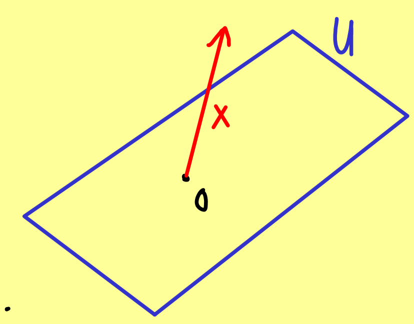
**Proof:** $\|x'\|_{X'} \geq \dfrac{|x'(x)|}{\|x\|_X} \Rightarrow 1 = \sup\limits_{\|x'\|=1} \|x'\|_{X'} \geq \sup\limits_{\|x'\|=1} \dfrac{|x'(x)|}{\|x\|_X}$

$\Rightarrow \|x\|_X \geq \sup\limits_{\|x'\|=1} |x'(x)|$

Use (a): $\|x\|_X \leq \sup\limits_{\|x'\|=1} |x'(x)|$

(d) Let $U \subseteq X$ be a closed subspace, $x \in X$ with $x \notin U$.

Then there exists $x' \in X'$ with $x'|_U = 0$ and $x'(x) \neq 0$.




**Proof:** $X/U := \{ [z] \mid z \in X \}$, $[z] := \{ z + u \mid u \in U \}$

$\|[z]\|_{X/U} := \inf\limits_{u \in U} \|z + u\|_X \rightsquigarrow (X/U, \|\cdot\|_{X/U})$ normed space

$\overset{(a)}{\Longrightarrow}$ There is a $y' \in (X/U)'$ with $y'([x]) \neq 0$.

Define $x' \in X'$ by $x'(z) := y'([z])$ for $z \in X$.

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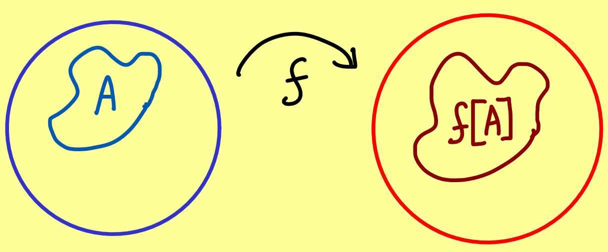
# Functional analysis – part 26

## Open mapping theorem (Banach–Schauder theorem)

What is an open map?



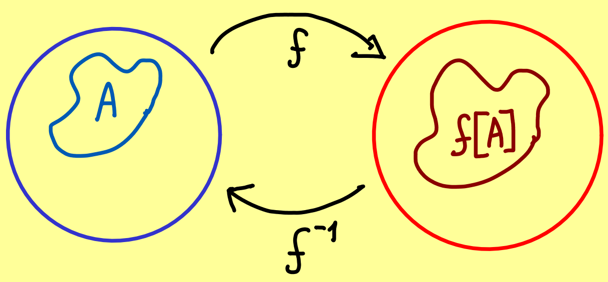
Let $(X, d_X)$, $(Y, d_Y)$ be two metric spaces.

$f: X \to Y$ is called open if

$$A \subseteq X \text{ open in } X \Rightarrow f[A] \subseteq Y \text{ open in } Y$$

General example: If $f: X \to Y$ is bijective and $f^{-1}: Y \to X$ is continuous, then:

$f: X \to Y$ is an open map



Continuity of $f^{-1}$: $A \subseteq X$ open in $X \Rightarrow \underbrace{(f^{-1})^{-1}[A]}_{f[A]} \subseteq Y$ open in $Y$

Examples: (a) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ open

(b) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ not open $\quad A = (-2,2) \rightsquigarrow f[A] = [0,4)$

Open Mapping Theorem: Let $X, Y$ be Banach spaces. For $T \in \mathcal{B}(X,Y)$ holds:

$$T \text{ surjective} \Leftrightarrow T \text{ open map}$$

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# Functional analysis – part 27

**Bounded inverse theorem:** $X, Y$ Banach spaces, $T \in \mathcal{B}(X,Y)$.

Then: $T$ bijective $\Rightarrow$ $T^{-1} \in \mathcal{B}(Y,X)$ (It's continuous)

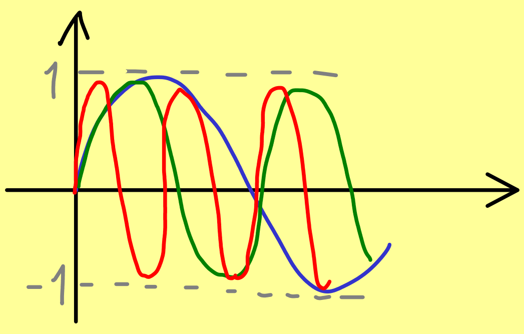
**Proof:** $T$ bijective $\underset{\text{open mapping theorem}}{\Rightarrow}$ $T$ open map $\Rightarrow$ $T^{-1}$ continuous $\square$

**Counterexample:** $X = \left( C([0,1]), \|\cdot\|_\infty \right)$, $Y = \left( \{ f \in C^1([0,1]) \mid f(0) = 0 \}, \|\cdot\|_\infty \right)$ $\rightarrow$ not complete

$$(Tf)(t) = \int_0^t f(s)\, ds$$ linear and bounded and bijective

$$\|Tf\|_\infty = \sup_{t \in [0,1]} \left| \int_0^t f(s)\, ds \right| \leq \|f\|_\infty \Rightarrow \|T\|_{X \to Y} \leq 1$$

Take $f_k(t) = \sin(kt)$




$$(Tf_k)(t) = \underbrace{\frac{1}{k}\left(1 - \cos(kt)\right)}_{g_k(t)}$$

$$T^{-1} g_k = f_k \Rightarrow \|T^{-1}\|_{Y \to X} \geq \frac{\|T^{-1} g_k\|_\infty}{\|g_k\|_\infty} = \frac{\overbrace{\|f_k\|_\infty}^{=1}}{\underbrace{\|g_k\|_\infty}_{\leq \frac{2}{k}}} \geq \frac{k}{2} \xrightarrow{k \to \infty} \infty$$

$\Rightarrow$ $T^{-1}$ not continuous

# Functional analysis – part 28

## Spectrum for bounded linear operators

Recall: $A \in \mathbb{C}^{n \times n}$ matrix with $n$ rows and $n$ columns.

$\lambda \in \mathbb{C}$ is called an eigenvalue of $A$ if:

$$\exists x \in \mathbb{C}^n \setminus \{0\} : \quad Ax = \lambda x$$

$$\Leftrightarrow \exists x \in \mathbb{C}^n \setminus \{0\} : \quad (A - \lambda I)x = 0$$

$$\Leftrightarrow \quad \mathrm{Ker}(A - \lambda I) \neq \{0\} \qquad \Leftrightarrow \quad \text{map } x \mapsto (A-\lambda I)x \text{ not injective}$$

Rank-nullity theorem: For any matrix $M \in \mathbb{C}^{m \times n}$:

$$\dim(\mathrm{Ran}(M)) + \dim(\mathrm{Ker}(M)) = n$$

Now: Let $X$ be a complex Banach space and $T: X \to X$ be a bounded linear operator.

Definition: The spectrum of $T$ is defined by: $\sigma(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not bijective} \}$

The resolvent set of $T$ is defined by: $\rho(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ bijective and } (T - \lambda I)^{-1} \text{ bounded} \}$

bounded inverse theorem

$$\Longrightarrow \qquad \sigma(T) = \mathbb{C} \setminus \rho(T)$$

We have the disjoint union: $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$

point spectrum $\quad \sigma_p(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not injective} \}$

continuous spectrum $\quad \sigma_c(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective with } \overline{\mathrm{Ran}(T - \lambda I)} = X \}$

residual spectrum $\quad \sigma_r(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective with } \overline{\mathrm{Ran}(T - \lambda I)} \neq X \}$

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## Functional analysis – part 29

Let $X$ be a complex Banach space and
$T: X \to X$ be a bounded linear operator.

$\lambda \in \sigma(T) \iff (T - \lambda)$ not invertible

Finite-dimensional example: $X = \mathbb{C}^n$, $Tx = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix}$

$\Rightarrow \sigma(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \sigma_p(T)$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors

Infinite-dimensional example: $X = \ell^p(\mathbb{N})$, $p \in [1, \infty)$

$$Tx = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \end{pmatrix}$$

Formally: For $\lambda_1, \lambda_2, \dots \in \mathbb{C}$ with $\sup_{j \in \mathbb{N}} |\lambda_j| < \infty$, define: $T: \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$
$(Tx)_j := \lambda_j x_j$

- $e_1 = (1, 0, 0, \dots)$ is an eigenvector with eigenvalue $\lambda_1$
- $e_2 = (0, 1, 0, \dots)$ is an eigenvector with eigenvalue $\lambda_2$
- $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\Rightarrow \sigma(T) \supseteq \{\lambda_1, \lambda_2, \dots\} = \sigma_p(T)$

Let $\mu \in \mathbb{C}$ be a number with $\mu \notin \{\lambda_1, \lambda_2, \dots\}$ but $\mu \in \overline{\{\lambda_1, \lambda_2, \dots\}}$.

e.g. $\lambda_j = \frac{1}{j}$ then $\mu = 0$

$\Rightarrow T - \mu$ is injective

Show: $T - \mu$ is not surjective

Proof: Assume $T - \mu$ is surjective $\Rightarrow T - \mu$ is bijective $\xRightarrow{\text{bounded inverse theorem}} (T - \mu)^{-1}$ bounded

$\Rightarrow \|(T-\mu)^{-1}\| \geq \|(T-\mu)^{-1} e_j\|_{\ell^p(\mathbb{N})} = \|(\lambda_j - \mu)^{-1} e_j\|_{\ell^p(\mathbb{N})} = |(\lambda_j - \mu)^{-1}|$

$= \frac{1}{|\lambda_j - \mu|} \xrightarrow{\text{for a subsequence}} \infty \quad ⚡$

Result: $\sigma(T) = \underbrace{\{\lambda_1, \lambda_2, \dots\}}_{\sigma_p(T)} \cup \underbrace{\{\mu \in \mathbb{C} \mid \mu \notin \{\lambda_1, \lambda_2, \dots\} \wedge \mu \in \overline{\{\lambda_1, \lambda_2, \dots\}}\}}_{\sigma_c(T) \; \cancel{\cup \; \sigma_r(T)}}$ $p \in [1, \infty)$

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## Functional analysis – part 30

$X$ complex Banach space

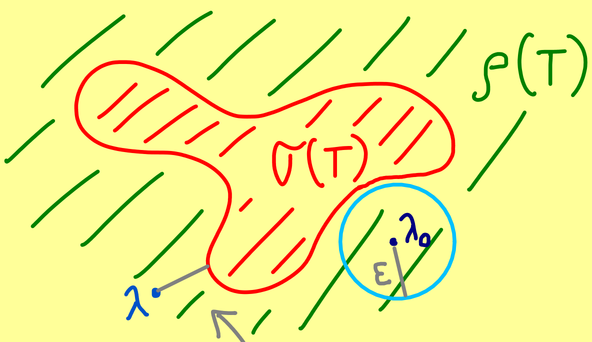
$T: X \to X$ bounded linear operator

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda) \text{ not invertible}\}$$

$$\rho(T) := \{\lambda \in \mathbb{C} \mid (T - \lambda) \text{ invertible}\}$$

**Proposition:** (a) $\rho(T)$ is an open set

$\sigma(T)$ is a closed set




(b) For $\lambda \in \rho(T)$: $\quad \|(T-\lambda)^{-1}\| \geq \dfrac{1}{\mathrm{dist}(\lambda, \sigma(T))}$

(c) The map $\rho(T) \longrightarrow \mathcal{B}(X)$

$\lambda \longmapsto (T-\lambda)^{-1}$ is analytical.

Locally, it can be expressed as a Taylor series.

**Proof:** Choose $\lambda_0 \in \rho(T)$ and set $C := \|(T-\lambda_0)^{-1}\|$, $\varepsilon := \dfrac{1}{C}$

Let's take any $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \varepsilon$.

Calculate: $T - \lambda = (T-\lambda_0) - (\lambda - \lambda_0) = (T-\lambda_0)\Big(I - \underbrace{(\lambda-\lambda_0)\cdot(T-\lambda_0)^{-1}}_{S}\Big)$

$\|S\| < \varepsilon \cdot C = 1$

> Neumann series: $(I - S)$ with $\|S\| < 1$ is invertible because
>
> $(I-S)\cdot\sum_{k=0}^{n} S^k = (I - S^{n+1}) \xrightarrow{n\to\infty} I \quad \Rightarrow (I-S)^{-1} = \sum_{k=0}^{\infty} S^k$

$\Longrightarrow$ $T - \lambda$ is invertible $\Longrightarrow \lambda \in \rho(T) \Longrightarrow \rho(T)$ is open (a) ✓

Also: $(T-\lambda)^{-1} = (I-S)^{-1}(T-\lambda_0)^{-1} = \sum_{k=0}^{\infty} S^k \cdot (T-\lambda_0)^{-1}$

$= \sum_{k=0}^{\infty} (\lambda-\lambda_0)^k \cdot (T-\lambda_0)^{-k} \cdot (T-\lambda_0)^{-1} = \sum_{k=0}^{\infty} (T-\lambda_0)^{-(k+1)} \cdot (\lambda-\lambda_0)^k$ (c) ✓

Now for $\lambda \in \sigma(T)$ $\overset{\text{above}}{\Longrightarrow}$ $|\lambda - \lambda_0| \geq \varepsilon \Longrightarrow \dfrac{1}{|\lambda-\lambda_0|} \leq C = \|(T-\lambda_0)^{-1}\|$

$\dfrac{1}{\mathrm{dist}(\lambda_0, \sigma(T))} = \dfrac{1}{\inf\limits_{\lambda\in\sigma(T)} |\lambda-\lambda_0|} = \sup\limits_{\lambda\in\sigma(T)} \dfrac{1}{|\lambda-\lambda_0|} \leq \|(T-\lambda_0)^{-1}\|$ (b) ✓

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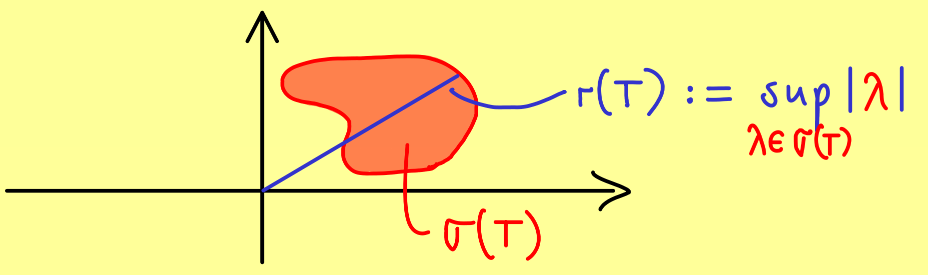
## Functional analysis – part 31

**Spectral radius:** $X$ complex Banach space $\quad T: X \to X$ bounded linear operator




**Theorem:** $X$ complex Banach space, $T: X \to X$ bounded linear operator.

Then: (a) $\sigma(T) \subseteq \mathbb{C}$ is compact

(b) $X \neq \{0\} \Rightarrow \sigma(T) \neq \emptyset$

(c) $r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{k \to \infty} \|T^k\|^{\frac{1}{k}} = \inf_{k \in \mathbb{N}} \|T^k\|^{\frac{1}{k}} \leq \|T\| < \infty$

**Proof:** For $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$: $\left(I - \frac{T}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$

$$\Rightarrow \boxed{(T - \lambda)^{-1} = -\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k \quad (*)}$$

$$\Rightarrow \sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\| \Rightarrow \sigma(T) \text{ is bounded}$$

**For (b):** Assume $\sigma(T) = \emptyset \Rightarrow \rho(T) = \mathbb{C}$

Reminder: The map $\rho(T) \longrightarrow \mathcal{B}(X)$

$\lambda \longmapsto (T - \lambda)^{-1}$ is analytic.

Take any $\ell \in \mathcal{B}(X)'$: $\quad f_\ell : \mathbb{C} \to \mathbb{C}$

$\lambda \mapsto \ell\left((T - \lambda)^{-1}\right)$

analytic function (holomorphic function)

We get that $f_\ell$ is a bounded entire function.

For $|\lambda| \geq 2 \cdot \|T\|$: $\quad \boxed{(T - \lambda)^{-1} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k \quad (*)}$

$$|f_\ell(\lambda)| \leq \|\ell\| \cdot \|(T - \lambda)^{-1}\| \leq \|\ell\| \underbrace{\frac{1}{|\lambda|}}_{\leq \frac{1}{2\|T\|}} \sum_{k=0}^{\infty} \underbrace{\left\|\frac{T}{\lambda}\right\|^k}_{\leq \frac{1}{2}}$$

$$\leq \frac{\|\ell\|}{\|T\|}$$

$$\xRightarrow{\text{Liouville's theorem}} f_\ell \text{ is constant}$$

$$f_\ell(0) = \ell(T^{-1})$$

$$\|$$

$$f_\ell(\lambda) = \ell\left((T - \lambda)^{-1}\right) = \ell\left(\sum_{k=0}^{\infty} (T)^{-(k+1)} \cdot (\lambda)^k\right)$$

$$= \sum_{k=0}^{\infty} \ell\left(T^{-(k+1)}\right) \cdot \lambda^k$$

$$\Rightarrow \ell(T^{-2}) = 0 \quad \text{for all } \ell \in \mathcal{B}(X)'$$

$$\xRightarrow[\text{(part 25)}]{\text{Hahn-Banach theorem}} T^{-2} = 0 \Rightarrow X = \{0\}$$

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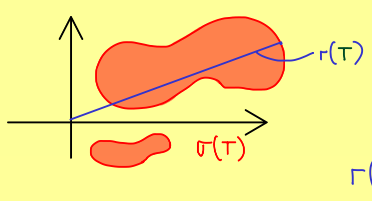
## Functional analysis – part 32




$T: X \to X$
bounded linear operator

$r(T) \leq \|T\|$

For normal operators: $r(T) = \|T\|$

$X$ is a complex Hilbert space

**Definition:** Let $X$ be a Hilbert space and $T: X \to X$ a bounded linear operator.
The bounded linear operator $T^*: X \to X$ defined by

$$\langle y, Tx \rangle = \langle T^* y, x \rangle \quad \text{for all } x, y \in X$$

is called the adjoint operator of $T$.

**Definition:** Let $X$ be a Hilbert space and $T: X \to X$ a bounded linear operator.
$T$ is called
(1) self-adjoint if $T^* = T$
(2) skew-adjoint if $T^* = -T$
(3) normal if $T^*T = TT^*$

**Proposition:** $T$ normal $\Rightarrow$ $r(T) = \|T\|$

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# Functional analysis – part 33

Compact operator: $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces.

$T: X \to Y$ bounded linear operator is called compact if

$\overline{T[B_1(0)]}$ is compact.

Example: matrix $A \in \mathbb{C}^{n \times n}$ (linear operator $\mathbb{C}^n \to \mathbb{C}^n$, $x \mapsto Ax$)

↪ compact

We know: $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ finite, non-empty set

$\mathrm{Ker}(A - \lambda_j)$ eigenspaces (finite-dimensional)

Proposition: $(X, \|\cdot\|_X)$ Banach space, $T: X \to X$ compact operator. Then:

(a) $\sigma(T)$ countable set (finite is possible)

(b) $\dim(X) = \infty \implies 0 \in \sigma(T)$

(c) $\sigma(T) \setminus \{0\}$ could be empty or finite.

Otherwise: $\sigma(T) \setminus \{0\} = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ ↙ no accumulation points other than 0

(d) Each $\lambda \in \sigma(T) \setminus \{0\}$ is an eigenvalue of $T$ $(\lambda \in \sigma_p(T))$

with $\dim(\mathrm{Ker}(T - \lambda)) < \infty$

Example: $X = \ell^2(\mathbb{N})$, $Tx = \left(\frac{1}{j} x_j\right)_{j \in \mathbb{N}}$

$$\overline{T[B_1(0)]} \subseteq \left\{ y \in \ell^2(\mathbb{N}) \;\middle|\; |y_j| \leq \frac{1}{j} \text{ for all } j \right\}$$

Hilbert cube
↓
compact set

$\implies T$ is a compact operator

$$T = \begin{pmatrix} \frac{1}{1} & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \\ & & & \frac{1}{4} & \\ & & & & \ddots \end{pmatrix}$$

$Te_k = \frac{1}{k} e_k$ (eigenvector) $\dim\left(\mathrm{Ker}\left(T - \frac{1}{k}\right)\right) = 1$

$$\sigma(T) = \left\{\frac{1}{1}, \frac{1}{2}, \ldots\right\} \cup \{0\}$$

# Functional Analysis – Part 34

## Spectral theorem of compact operators

Let $X$ be a complex Hilbert space and $T: X \to X$ be a compact operator. Assume that $T$ is self-adjoint ($T^* = T$) or normal ($T^*T = TT^*$).

Then there is an orthonormal system $\{e_i \mid i \in I\}$ with $I \subseteq \mathbb{N}$ and a sequence $(\lambda_i)_{i\in I}$ in $\mathbb{C} \setminus \{0\}$ with $\lambda_i \xrightarrow{i\to\infty} 0$ (if $I$ infinite)

such that:

$$X = \operatorname{Ker}(T) \oplus^{\perp} \overline{\operatorname{Span}(e_i \mid i \in I)}$$

orthogonal sum: $X = U \oplus^{\perp} V$ means:

for each $x \in X$ there is $u \in U$, $v \in V$:

- $x = u + v$
- $u \perp v$

unique!

and $$Tx = \sum_{k\in I} \lambda_k e_k \langle e_k, x\rangle \quad \text{for } x \in X$$

eigenvector to $\lambda_k$

eigenvalue

and $$\|T\| = \sup_{k\in I} |\lambda_k| .$$