The Bright Side of Mathematics

Functional analysis – part 33

<u>Compact operator:</u> $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces.

$T: X \to Y$ bounded linear operator is called <u>compact</u> if

$\overline{T[B_1(0)]}$ is compact.

<u>Example:</u> matrix $A \in \mathbb{C}^{n \times n}$ (linear operator $\mathbb{C}^n \to \mathbb{C}^n$, $x \mapsto Ax$)

↪ compact

We know: $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ finite, non-empty set

$\ker(A - \lambda_j)$ eigenspaces (finite-dimensional)

<u>Proposition:</u> $(X, \|\cdot\|_X)$ Banach space, $T: X \to X$ compact operator. Then:

(a) $\sigma(T)$ countable set (finite is possible)

(b) $\dim(X) = \infty \Rightarrow 0 \in \sigma(T)$

(c) $\sigma(T) \setminus \{0\}$ could be empty or finite.

Otherwise: $\sigma(T) \setminus \{0\} = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ ↙ no accumulation points other than 0

(d) Each $\lambda \in \sigma(T) \setminus \{0\}$ is an eigenvalue of T $(\lambda \in \sigma_p(T))$

with $\dim(\ker(T - \lambda)) < \infty$

<u>Example:</u> $X = \ell^2(\mathbb{N})$, $Tx = \left(\frac{1}{j} x_j\right)_{j \in \mathbb{N}}$

$$\overline{T[B_1(0)]} \subseteq \left\{ y \in \ell^2(\mathbb{N}) \;\middle|\; |y_j| \leq \frac{1}{j} \text{ for all } j \right\}$$

Hilbert cube
↓
compact set

$\Rightarrow T$ is a compact operator

$$T = \begin{pmatrix} \frac{1}{1} & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \\ & & & \frac{1}{4} & \\ & & & & \ddots \end{pmatrix}$$

$Te_k = \frac{1}{k} e_k$ (eigenvector) $\quad \dim\left(\ker\left(T - \frac{1}{k}\right)\right) = 1$

$\sigma(T) = \left\{\frac{1}{1}, \frac{1}{2}, \ldots\right\} \cup \{0\}$