BECOME A MEMBER

ON STEADY

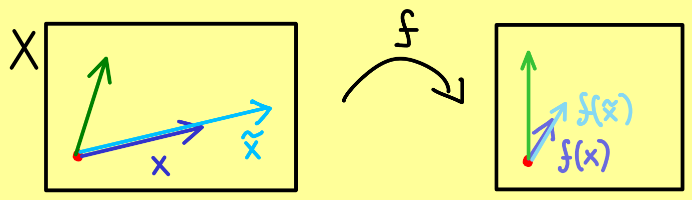
The Bright Side of Mathematics

Functional analysis – part 21

Isomorphisms?

Homomorphism : map that preserves structures

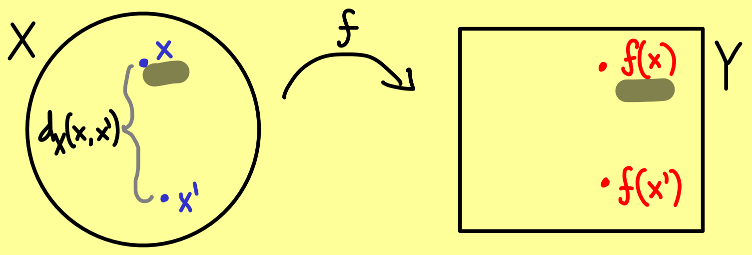
Example: (a) Let X, Y be vector spaces and $f: X \to Y$ be a map.



$$\left.\begin{aligned} f(\lambda \cdot x) &= \lambda \cdot f(x) \\ f(x + x') &= f(x) + f(x') \end{aligned}\right\} \text{linear}$$

homomorphism = linear map

(b) Let (X, d_X), (Y, d_Y) be two metric spaces and $f: X \to Y$ be a map.



$$d_Y(f(x), f(x')) \leq d_X(x, x') \quad (*)$$

homomorphism = map that satisfies $(*)$

isomorphism = homomorphism + bijective + inverse map is also homomorphism

Isomorphism for Banach spaces X, Y:

$f: X \to Y$ with: linear + bijective + $\|f(x)\|_Y = \|x\|_X$

(often called isometric isomorphism)

Example: (a) $S_R: \ell^p(\mathbb{N}) \longrightarrow \ell^p(\mathbb{N})$, $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$

$\Rightarrow$ linear, $\|S_R x\|_p = \|x\|_p$ not surjective $\Rightarrow$ not an isomorphism

(b) $S: \ell^p(\mathbb{Z}) \longrightarrow \ell^p(\mathbb{Z})$, $(\dots, x_{-1}, x_0, x_1, x_2, \dots) \mapsto (\dots, x_{-2}, x_{-1}, x_0, x_1, \dots)$

index $\dots$ -1 0 1 2 $\dots$ index $\dots$ -1 0 1 2 $\dots$

$\Rightarrow$ linear, $\|S x\|_p = \|x\|_p$ and bijective $\Rightarrow$ isomorphism