The Bright Side of Mathematics

Functional analysis - part 9

Examples of Hilbert spaces

(a) $\mathbb{R}^n$, $\mathbb{C}^n$ with $\langle x, y \rangle = \sum_{i=1}^{n} \overline{x_i}\, y_i$

(b) $\ell^2(\mathbb{N}, \mathbb{F})$ with $\langle x, y \rangle = \sum_{i=1}^{\infty} \overline{x_i}\, y_i$

Not a Hilbert space ⟶ (c) $C([0,1], \mathbb{F})$ with $\langle f, g \rangle = \int_0^1 \overline{f(t)}\, g(t)\, dt$ ↙ inner product

$(\ell^2(\mathbb{N}, \mathbb{F}), \langle \cdot, \cdot \rangle)$ is a Hilbert space: $\langle \cdot, \cdot \rangle : \ell^2 \times \ell^2 \longrightarrow \mathbb{F}$ ↗ later!

(1) positive definite: $\langle x, x \rangle = \sum_{i=1}^{\infty} \overline{x_i}\, x_i = \sum_{i=1}^{\infty} |x_i|^2 \geq 0$

and $\langle x, x \rangle = 0 \Rightarrow |x_i|^2 = 0$ for all $i \in \mathbb{N}$

$\Rightarrow x_i = 0$ for all $i \in \mathbb{N}$ $\Rightarrow x = 0$.

(2) (conjugate) symmetric: $\overline{\langle y, x \rangle} = \sum_{i=1}^{\infty} \overline{\overline{y_i}\, x_i} = \sum_{i=1}^{\infty} y_i\, \overline{x_i} = \langle x, y \rangle$

(3) linear in the 2^{nd} argument: $\langle x, y + z \rangle = \sum_{i=1}^{\infty} \overline{x_i}(y_i + z_i) = \sum_{i=1}^{\infty} \overline{x_i}\, y_i + \sum_{i=1}^{\infty} \overline{x_i}\, z_i$

$= \langle x, y \rangle + \langle x, z \rangle$

$\langle x, \lambda \cdot y \rangle = \sum_{i=1}^{\infty} \overline{x_i}(\lambda y_i) = \lambda \cdot \sum_{i=1}^{\infty} \overline{x_i}\, y_i = \lambda \cdot \langle x, y \rangle$