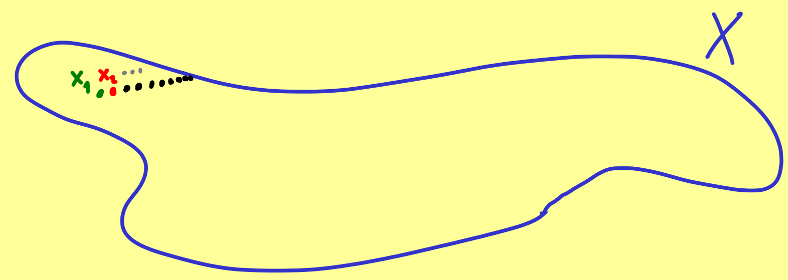
The Bright Side of
Mathematics

Functional analysis - part 4

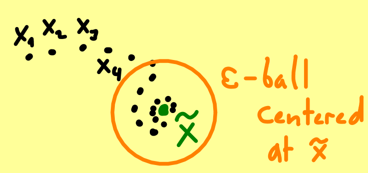
(X, d) metric space



Sequence in X: $(x_1, x_2, x_3, \dots)$ or $(x_n)_{n\in\mathbb{N}}$ or $x: \mathbb{N} \longrightarrow X$, $n \longmapsto x_n$ map

Convergence: A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is called convergent if there is $\tilde{x} \in X$ with

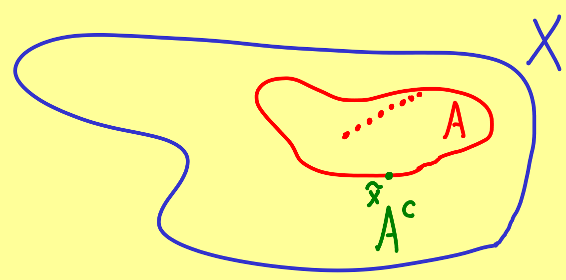
$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N : \quad d(x_n, \tilde{x}) < \varepsilon .$$



We write: $x_n \xrightarrow{n\to\infty} \tilde{x}$ or $\lim\limits_{n\to\infty} x_n = \tilde{x}$.

Proposition: $A \subseteq X$ is closed

$\Longleftrightarrow$ For every convergent sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$, one has $\lim\limits_{n\to\infty} a_n \in A$



Proof: $(\Leftarrow)$: Show it by contraposition! Assume A is not closed.

$\Rightarrow$ $A^c := X \setminus A$ is not open.

$\Rightarrow$ There is an $\tilde{x} \in A^c$ with $B_\varepsilon(\tilde{x}) \cap A \neq \emptyset$ for all $\varepsilon > 0$.

$\Rightarrow$ There is a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in B_{\frac{1}{n}}(\tilde{x}) \cap A$

$\Rightarrow$ $\lim\limits_{n\to\infty} a_n = \tilde{x} \notin A$

$(\Rightarrow)$: Show it by contraposition! Assume there is $(a_n)_{n\in\mathbb{N}} \subseteq A$ with $\tilde{x} := \lim\limits_{n\to\infty} a_n \notin A$.

$\Rightarrow$ $B_\varepsilon(\tilde{x}) \cap A \neq \emptyset$ for all $\varepsilon > 0$. $\Rightarrow$ A^c is not open $\Rightarrow$ A is not closed