

Fourier Transform - Part 11

Let's prove:
$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}, \quad x \in [0, 2\pi]$$

Note:

$$\begin{aligned} \frac{1}{2} + \sum_{k=1}^n \cos(kx) &= \frac{1}{2} + \sum_{k=1}^n \frac{1}{2} \cdot (e^{ikx} + e^{-ikx}) = \frac{1}{2} \sum_{k=-n}^n e^{ikx} \\ &= \frac{1}{2} e^{-inx} \sum_{k=0}^{2n} \underbrace{e^{ikx}}_{q^k} \quad \leftarrow q = e^{ix} \\ &= \frac{1}{2} e^{-inx} \cdot \frac{1 - q^{2n+1}}{1 - q} \quad \leftarrow \text{geometric sum formula } q \neq 1 \\ &= \frac{1}{2} \frac{e^{-inx} - e^{i(n+1)x}}{1 - e^{ix}} \cdot \frac{-e^{-\frac{1}{2}ix}}{-e^{-\frac{1}{2}ix}} \\ &= \frac{1}{2} \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}} \cdot \frac{\frac{1}{2i}}{\frac{1}{2i}} = \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{1}{2}x)} \end{aligned}$$

for $x \in \mathbb{R} \setminus \{2\pi m \mid m \in \mathbb{Z}\}$

Lemma:
$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2} \quad \text{for } x \in (0, 2\pi)$$

and we have uniform convergence on interval $[\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$.

Proof:

$$\begin{aligned} \sum_{k=1}^n \frac{\sin(kx)}{k} &= \sum_{k=1}^n \int_{\pi}^x \cos(kt) dt = \int_{\pi}^x \sum_{k=1}^n \cos(kt) dt \\ &= \int_{\pi}^x \left(\frac{1}{2} \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} - \frac{1}{2} \right) dt \\ &= \underbrace{\int_{\pi}^x \frac{\sin((n+\frac{1}{2})t)}{2 \sin(\frac{1}{2}t)} dt}_{f_n(x)} - \frac{1}{2}(x - \pi) \end{aligned}$$

integration by parts: $f_n(x) = \int_{\pi}^x \underbrace{\frac{1}{2 \sin(\frac{1}{2}t)}}_u \cdot \underbrace{\sin((n+\frac{1}{2})t)}_{v'} dt$

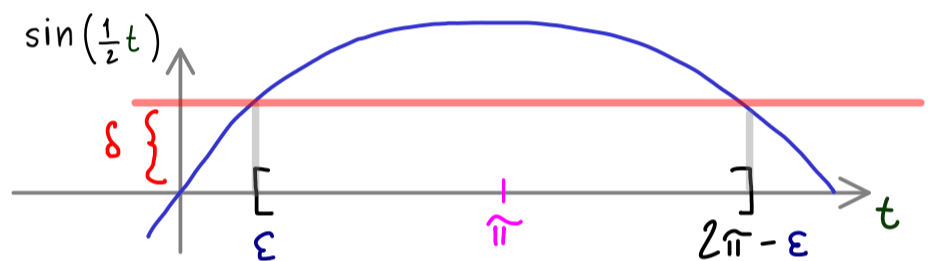
$u' = -\frac{1}{2} \frac{\cos(\frac{1}{2}t)}{(\sin(\frac{1}{2}t))^2}$

$v = \frac{1}{n+\frac{1}{2}} \cdot (-1) \cdot \cos((n+\frac{1}{2})t)$

$$f_n(x) = \frac{1}{n+\frac{1}{2}} \cdot \frac{(-1) \cos((n+\frac{1}{2})t)}{2 \sin(\frac{1}{2}t)} \Big|_{\pi}^x - \int_{\pi}^x \frac{1}{n+\frac{1}{2}} \frac{(-1) \cdot \cos((n+\frac{1}{2})t) \cos(\frac{1}{2}t)}{(-4) \cdot (\sin(\frac{1}{2}t))^2} dt$$

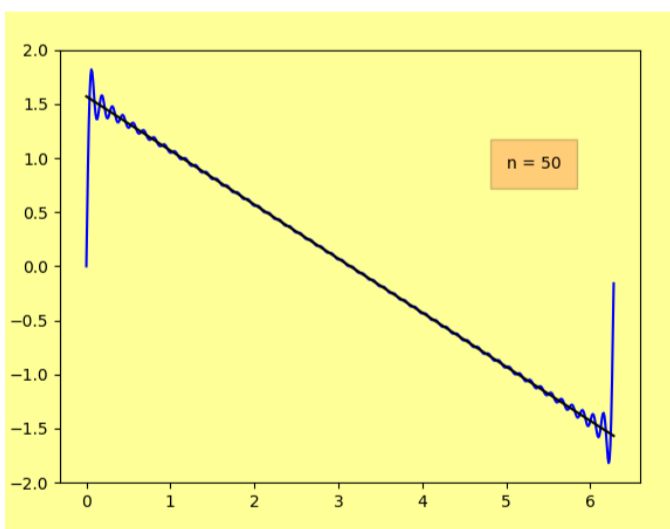
$$= \frac{1}{n+\frac{1}{2}} \left(\underbrace{\frac{(-1) \cos((n+\frac{1}{2})x)}{2 \sin(\frac{1}{2}x)}}_{= a(x)} - \frac{1}{4} \int_{\pi}^x \underbrace{\frac{\cos((n+\frac{1}{2})t) \cos(\frac{1}{2}t)}{(\sin(\frac{1}{2}t))^2}}_{b(x)} dt \right)$$

For $\epsilon > 0$, choose $x \in [\epsilon, 2\pi - \epsilon]$:



$$\|f_n\|_{\infty} \leq \frac{1}{n+\frac{1}{2}} \left(\|a\|_{\infty} + \|b\|_{\infty} \right)$$

$$\leq \frac{1}{n+\frac{1}{2}} \left(\frac{1}{2\delta} + \frac{1}{4\delta^2} \cdot \pi \right) \xrightarrow{n \rightarrow \infty} 0$$



Recall $f_n(x) = \sum_{k=1}^n \frac{\sin(kx)}{k} + \frac{1}{2}(x - \pi)$

□

Theorem: $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}, \quad x \in [0, 2\pi]$

uniform convergence on $[0, 2\pi]$

Proof: For $\varepsilon > 0$, $x, x_0 \in [\varepsilon, 2\pi - \varepsilon]$: (use Lemma)

$$\int_{x_0}^x \sum_{k=1}^{\infty} \frac{\sin(kt)}{k} dt = \int_{x_0}^x \frac{\pi - t}{2} dt = -\frac{(\pi - t)^2}{4} \Big|_{x_0}^x = -\frac{(x - \pi)^2}{4} + \underbrace{\frac{(x_0 - \pi)^2}{4}}_{C_0}$$

uniform convergence \rightarrow //

$$\sum_{k=1}^{\infty} \int_{x_0}^x \frac{\sin(kt)}{k} dt = \sum_{k=1}^{\infty} -\frac{\cos(kt)}{k^2} \Big|_{x_0}^x = -\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} + C_1$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x - \pi)^2}{4} + C \quad \leftarrow \text{calculate it!}$$

\rightarrow still uniform convergence on $[\varepsilon, 2\pi - \varepsilon]$

We know more:

(1) $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$ uniformly convergent on $[0, 2\pi]$

by Weierstrass M-test since $\left| \frac{\cos(kx)}{k^2} \right| \leq \frac{1}{k^2}$

$$\Rightarrow [0, 2\pi] \ni x \mapsto \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} \text{ continuous function}$$

(2) $[0, 2\pi] \ni x \mapsto \frac{(x - \pi)^2}{4} + C$ continuous function

(3) $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x - \pi)^2}{4} + C$ for all $x \in (0, 2\pi)$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x - \pi)^2}{4} + C \text{ uniformly convergent on } [0, 2\pi]$$

Find C : $\int_0^{2\pi} \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} dx = \int_0^{2\pi} \left(\frac{(x - \pi)^2}{4} + C \right) dx = \underbrace{\frac{(x - \pi)^3}{12} \Big|_0^{2\pi}}_{\frac{\pi^3}{6}} + 2\pi \cdot C$

// uniform convergence

$$\sum_{k=1}^{\infty} \int_0^{2\pi} \frac{\cos(kx)}{k^2} dx = 0 \quad \Rightarrow \quad C = -\frac{\pi^2}{12}$$