# Fourier Transform - Part 11

Let's prove: $$\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12} \quad , \quad x \in [0, 2\pi]$$

Note:
$$\frac{1}{2} + \sum_{k=1}^{n} \cos(kx)) = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{2} \cdot \left( e^{ikx} + e^{-ikx} \right) = \frac{1}{2} \sum_{k=-n}^{n} e^{ikx}$$

$$= \frac{1}{2} e^{-inx} \sum_{k=0}^{2n} \underbrace{e^{ikx}}_{q^k} \qquad q = e^{ix}$$

$$= \frac{1}{2} e^{-inx} \cdot \frac{1 - q^{2n+1}}{1 - q}$$ geometric sum formula $q \neq 1$

$$= \frac{1}{2} \frac{e^{-inx} - e^{i(n+1)x}}{1 - e^{ix}} \cdot \frac{-e^{-\frac{1}{2}ix}}{-e^{-\frac{1}{2}ix}}$$

$$= \frac{1}{2} \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}} \cdot \frac{\frac{1}{2i}}{\frac{1}{2i}} = \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$$

for $x \in \mathbb{R} \setminus \{2\pi m \mid m \in \mathbb{Z}\}$

Lemma: $$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2} \quad \text{for } x \in (0, 2\pi)$$

and we have uniform convergence on interval $[\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$.

Proof:
$$\sum_{k=1}^{n} \frac{\sin(kx)}{k} = \sum_{k=1}^{n} \int_{\pi}^{x} \cos(kt))\,dt = \int_{\pi}^{x} \sum_{k=1}^{n} \cos(kt))\,dt$$

$$= \int_{\pi}^{x} \left( \frac{1}{2} \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} - \frac{1}{2} \right) dt$$

$$= \underbrace{\int_{\pi}^{x} \frac{\sin((n+\frac{1}{2})t)}{2\sin(\frac{1}{2}t)}\,dt}_{f_n(x)} - \frac{1}{2}(x - \pi)$$

integration by parts: $$f_n(x) = \int_\pi^x \underbrace{\frac{1}{2\sin\left(\frac12 t\right)}}_{u} \cdot \underbrace{\sin\left(\left(n+\tfrac12\right)t\right)}_{v'}\, dt$$

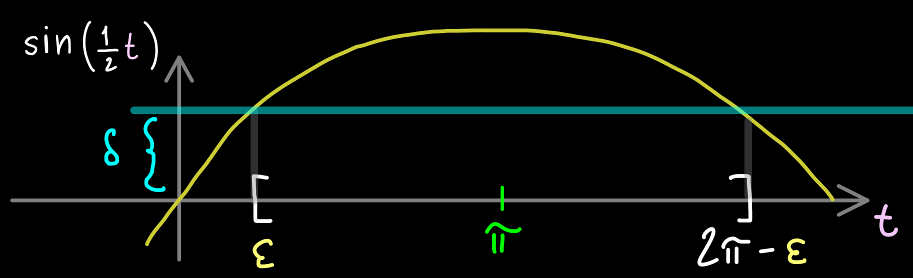
$$u' = -\frac12 \, \frac{\frac12 \cos\left(\frac12 t\right)}{\left(\sin\left(\frac12 t\right)\right)^2}$$

$$v = \frac{1}{n+\frac12} \cdot (-1)\cdot \cos\left(\left(n+\tfrac12\right)t\right)$$

$$f_n(x) = \frac{1}{n+\frac12} \cdot \frac{(-1)\cos\left(\left(n+\frac12\right)t\right)}{2\sin\left(\frac12 t\right)}\Bigg|_\pi^x - \int_\pi^x \frac{1}{n+\frac12}\, \frac{(-1)\cdot\cos\left(\left(n+\frac12\right)t\right)\cos\left(\frac12 t\right)}{(-4)\cdot\left(\sin\left(\frac12 t\right)\right)^2}\, dt$$

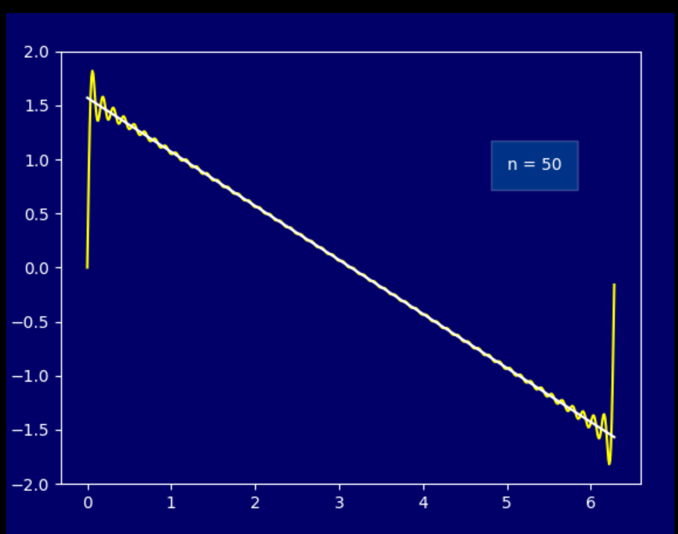
$$= \frac{1}{n+\frac12}\left( \underbrace{\frac{(-1)\cos\left(\left(n+\frac12\right)x\right)}{2\sin\left(\frac12 x\right)}}_{=a(x)} - \underbrace{\frac14 \int_\pi^x \frac{\cos\left(\left(n+\frac12\right)t\right)\cos\left(\frac12 t\right)}{\left(\sin\left(\frac12 t\right)\right)^2}\, dt}_{b(x)} \right)$$

For $\varepsilon > 0$, choose $x \in [\varepsilon, 2\pi - \varepsilon]$:




$$\| f_n \|_\infty \le \frac{1}{n+\frac12}\left( \|a\|_\infty + \|b\|_\infty \right)$$

$$\le \frac{1}{n+\frac12}\left( \frac{1}{2\delta} + \frac{1}{4\delta^2}\cdot \pi \right) \xrightarrow{n\to\infty} 0$$




Recall $$f_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k} + \frac12 (x - \pi)$$

$\square$

Theorem: $$\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12} \quad , \quad x \in [0, 2\pi]$$

uniform convergence on $[0, 2\pi]$

**Proof:** For $\varepsilon > 0,\ x, x_0 \in [\varepsilon, 2\pi - \varepsilon]$: (use Lemma)

$$\int_{x_0}^{x} \sum_{k=1}^{\infty} \frac{\sin(kt)}{k}\, dt = \int_{x_0}^{x} \frac{\pi - t}{2}\, dt = -\frac{(\pi - t)^2}{4}\Big|_{x_0}^{x} = -\frac{(x-\pi)^2}{4} + \underbrace{\frac{(x_0-\pi)^2}{4}}_{C_0}$$

uniform convergence $\rightarrow$ $\parallel$

$$\sum_{k=1}^{\infty} \int_{x_0}^{x} \frac{\sin(kt)}{k}\, dt = \sum_{k=1}^{\infty} -\frac{\cos(kt))}{k^2}\Big|_{x_0}^{x} = -\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} + C_1$$

$$\Longrightarrow \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C$$

calculate it!

still uniform convergence on $[\varepsilon, 2\pi - \varepsilon]$

**We know more:**

(1) $\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}$ uniformly convergent on $[0, 2\pi]$

by Weierstrass M-test since $\left|\frac{\cos(kx))}{k^2}\right| \leq \frac{1}{k^2}$

$\Longrightarrow [0,2\pi] \ni x \mapsto \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}$ continuous function

(2) $[0,2\pi] \ni x \mapsto \frac{(x-\pi)^2}{4} + C$ continuous function

(3) $\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C$ for all $x \in (0, 2\pi)$

$\Longrightarrow \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C$ uniformly convergent on $[0, 2\pi]$

Find $C$:

$$\int_0^{2\pi} \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}\, dx = \int_0^{2\pi} \left(\frac{(x-\pi)^2}{4} + C\right) dx = \underbrace{\frac{(x-\pi)^3}{12}\Big|_0^{2\pi}}_{\frac{\pi^3}{6}} + 2\pi \cdot C$$

$\parallel$ uniform convergence

$$\sum_{k=1}^{\infty} \int_0^{2\pi} \frac{\cos(kx))}{k^2}\, dx = 0 \qquad \Longrightarrow C = -\frac{\pi^2}{12}$$