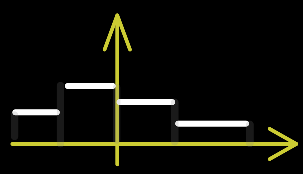
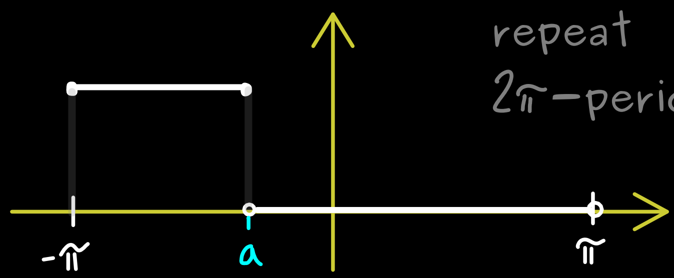
Fourier Transform – Part 10

For proving Parseval's identity $\rightsquigarrow$ step functions



Most important step function:

$$h_a(x) = \begin{cases} 1, & x \in [-\pi, a] \\ 0, & x \in (a, \pi) \end{cases}$$



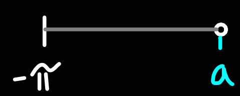
repeat 2π-periodically

for every $a \in [-\pi, \pi]$

Fourier series for this example:

$$c_k = \langle e_k, h_a \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} h_a(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{a} e^{-ikx}\, dx$$

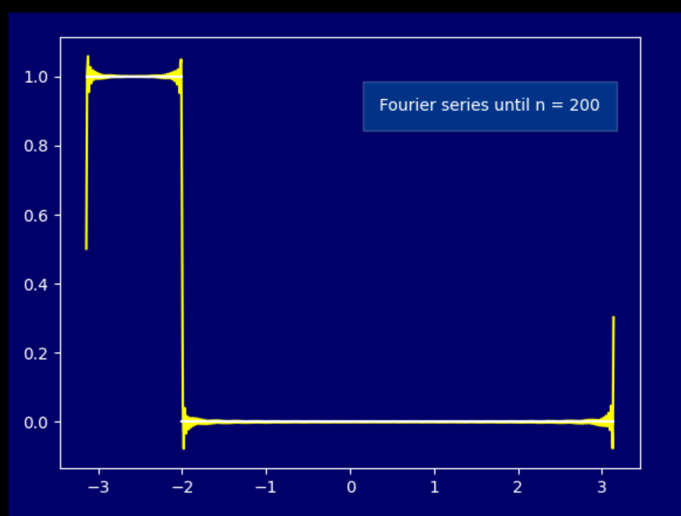
$$= \begin{cases} \dfrac{a+\pi}{2\pi}, & k = 0 \\ \dfrac{1}{2\pi(-ik)} \left(e^{-ika} - e^{ik\pi} \right), & k \neq 0 \end{cases}$$



Visualization:

$$a_k = 2 \cdot \mathrm{Re}(c_k)$$

$$b_k = -2 \cdot \mathrm{Im}(c_k)$$



Show Parseval's identity:

$$k \neq 0: \quad |c_k|^2 = \frac{1}{2\pi(-ik)}\left(e^{-ika} - e^{ik\pi}\right)\overline{\frac{1}{2\pi(-ik)}\left(e^{-ika} - e^{ik\pi}\right)}$$

$$= \frac{1}{4\pi^2 k^2}\cdot\left(e^{-ika} - e^{ik\pi}\right)\cdot\left(e^{ika} - e^{-ik\pi}\right)$$

$$= \frac{1}{4\pi^2 k^2}\cdot\left(1 - e^{ik(\pi+a)} - e^{-ik(\pi+a)} + 1\right)$$

$$= \frac{1}{4\pi^2 k^2}\cdot\left(2 - 2\cos(k(\pi+a))\right) = \frac{1}{2\pi^2 k^2}\cdot\left(1 - \cos(k(\pi+a))\right)$$

$$\Rightarrow \sum_{k=-n}^{n} |c_k|^2 = \left(\frac{a+\pi}{2\pi}\right)^2 + \frac{1}{2\pi^2}\left(\sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{k^2} - \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{\cos(k(\pi+a))}{k^2}\right)$$

$$= \left(\frac{a+\pi}{2\pi}\right)^2 + \frac{1}{\pi^2}\left(\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{\cos(k(\pi+a))}{k^2}\right)$$

$\xrightarrow{n\to\infty}$ (*) $= \frac{\pi^2}{6}$ $\qquad$ $\xrightarrow{n\to\infty}$ (**) $\frac{a^2}{4} - \frac{\pi^2}{12}$

General formula: $x \in [0, 2\pi]$

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 = \left(\frac{a+\pi}{2\pi}\right)^2 + \frac{1}{\pi^2}\left(\frac{\pi^2}{6} - \frac{a^2}{4} + \frac{\pi^2}{12}\right)$$

$$= \left(\frac{a+\pi}{2\pi}\right)^2 + \frac{1}{4} - \frac{a^2}{4\pi^2} = \frac{2a\pi + \pi^2}{4\pi^2} + \frac{1}{4}$$

$$= \frac{a}{2\pi} + \frac{1}{2} = \frac{1}{2\pi}\cdot(a+\pi) = \frac{1}{2\pi}\int_{-\pi}^{a} 1\,dx = \langle h_a, h_a \rangle$$

$$= \|h_a\|^2$$