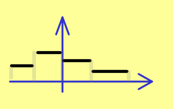
BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

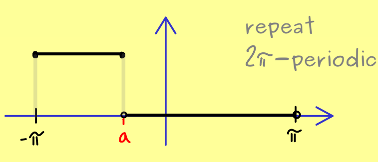
Fourier Transform – Part 10

For proving Parseval's identity ⇝ step functions



Most important step function:

$$h_a(x) = \begin{cases} 1, & x \in [-\pi, a] \\ 0, & x \in (a, \pi) \end{cases}$$



repeat 2π-periodically

for every $a \in [-\pi, \pi]$

Fourier series for this example:

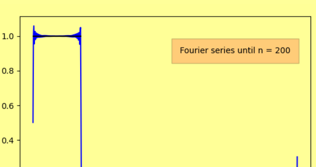
$$c_k = \langle e_k, h_a \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} h_a(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{a} e^{-ikx}\, dx$$

$$= \begin{cases} \frac{a+\pi}{2\pi}, & k = 0 \\ \frac{1}{2\pi(-ik)} \left(e^{-ika} - e^{ik\pi} \right), & k \neq 0 \end{cases}$$

Visualization:

$$a_k = 2 \cdot \mathrm{Re}(c_k)$$

$$b_k = -2 \cdot \mathrm{Im}(c_k)$$



Show Parseval's identity:

$k \neq 0$:

$$|c_k|^2 = \frac{1}{2\pi(-ik)} \left(e^{-ika} - e^{ik\pi} \right) \overline{\frac{1}{2\pi(-ik)} \left(e^{-ika} - e^{ik\pi} \right)}$$

$$= \frac{1}{4\pi^2 k^2} \cdot \left(e^{-ika} - e^{ik\pi} \right) \cdot \left(e^{ika} - e^{-ik\pi} \right)$$

$$= \frac{1}{4\pi^2 k^2} \cdot \left(1 - e^{ik(\pi + a)} - e^{-ik(\pi + a)} + 1 \right)$$

$$= \frac{1}{4\pi^2 k^2} \cdot \left(2 - 2\cos(k(\pi + a)) \right) = \frac{1}{2\pi^2 k^2} \cdot \left(1 - \cos(k(\pi + a)) \right)$$

$$\Rightarrow \sum_{k=-n}^{n} |c_k|^2 = \left(\frac{a+\pi}{2\pi} \right)^2 + \frac{1}{2\pi^2} \left(\sum_{\substack{k=-n \\ k \neq 0}}^{n} \frac{1}{k^2} - \sum_{\substack{k=-n \\ k \neq 0}}^{n} \frac{\cos(k(\pi + a))}{k^2} \right)$$

$$= \left(\frac{a+\pi}{2\pi} \right)^2 + \frac{1}{\pi^2} \left(\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{\cos(k(\pi + a))}{k^2} \right)$$

$n \to \infty$: (*) $= \frac{\pi^2}{6}$; $n \to \infty$: (**) $\frac{a^2}{4} - \frac{\pi^2}{12}$

General formula: $x \in [0, 2\pi]$

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 = \left(\frac{a+\pi}{2\pi} \right)^2 + \frac{1}{\pi^2} \left(\frac{\pi^2}{6} - \frac{a^2}{4} + \frac{\pi^2}{12} \right)$$

$$= \left(\frac{a+\pi}{2\pi} \right)^2 + \frac{1}{4} - \frac{a^2}{4\pi^2} = \frac{2a\pi + \pi^2}{4\pi^2} + \frac{1}{4}$$

$$= \frac{a}{2\pi} + \frac{1}{2} = \frac{1}{2\pi} \cdot (a + \pi) = \frac{1}{2\pi} \int_{-\pi}^{a} 1\, dx = \langle h_a, h_a \rangle$$

$$= \| h_a \|^2$$