



Fourier Transform - Part 9

$L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$ has ONS $(\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots)$ given by $e_k: x \mapsto e^{ikx}$

\rightsquigarrow Fourier series $\mathcal{F}_n(f) = \sum_{k=-n}^n e_k \langle e_k, f \rangle$

Parseval's identity: $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, f \rangle|^2$

$$\Leftrightarrow \|f - \mathcal{F}_n(f)\| \xrightarrow{n \rightarrow \infty} 0$$

means: $f = \mathcal{F}_n(f) + r_n$ with $\|r_n\| \xrightarrow{n \rightarrow \infty} 0$

Consider two functions: $f, g \in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$

$\langle f, g \rangle \leftarrow$ formula with Fourier coefficients?

$$f = \mathcal{F}_n(f) + r_n \quad \text{with} \quad \|r_n\| \xrightarrow{n \rightarrow \infty} 0$$

$$g = \mathcal{F}_n(g) + \tilde{r}_n \quad \text{with} \quad \|\tilde{r}_n\| \xrightarrow{n \rightarrow \infty} 0$$

$$\text{We have: } |\langle \mathcal{F}_n(g), r_n \rangle| \leq \|\mathcal{F}_n(g)\| \|r_n\|$$

Cauchy
Schwarz

$$\leq \|g\| \cdot \|r_n\| \xrightarrow{n \rightarrow \infty} 0$$

Bessel's inequality \nearrow

$$\langle f, g \rangle = \langle \mathcal{F}_n(f) + r_n, \mathcal{F}_n(g) + \tilde{r}_n \rangle$$

$$= \langle \mathcal{F}_n(f), \mathcal{F}_n(g) \rangle + \underbrace{\langle r_n, \mathcal{F}_n(g) \rangle + \langle \mathcal{F}_n(f), \tilde{r}_n \rangle}_{(*)} + \langle r_n, \tilde{r}_n \rangle$$

$$= \left\langle \sum_{k=-n}^n e_k \langle e_k, f \rangle, \sum_{l=-n}^n e_l \langle e_l, g \rangle \right\rangle + (*)$$

$$\begin{aligned}
&= \sum_{k=-n}^n \sum_{l=-n}^n \overline{\langle e_k, f \rangle} \langle e_l, g \rangle \underbrace{\langle e_k, e_l \rangle}_{=\delta_{kl}} + (*) \\
&= \sum_{k=-n}^n \langle f, e_k \rangle \langle e_k, g \rangle + (*) \\
&\xrightarrow{h \rightarrow \infty} \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle
\end{aligned}$$

Remember the equivalent statements: $L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$ with ONS $(e_k)_{k \in \mathbb{Z}}$

(a) Parseval's identity: $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, f \rangle|^2$

(b) ONS is complete: $\left\| f - \sum_{k=-n}^n e_k \langle e_k, f \rangle \right\| \xrightarrow{h \rightarrow \infty} 0$
 $\left(f = \sum_{k=-\infty}^{\infty} e_k \langle e_k, f \rangle \right)$

(c) ONS gives inner product:

$$\langle f, g \rangle = \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle \quad \left(\sum_{k=-\infty}^{\infty} |e_k\rangle \langle e_k| = \mathbb{1} \right) \quad \text{informal:}$$

(d) ONS is total: $\text{span}\left(\{e_k\}_{k \in \mathbb{Z}}\right)$ is dense in $L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$:

$$\forall f \in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C}) \quad \forall \epsilon > 0 \quad \exists N \in \mathbb{N}, \lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{C}:$$

$$\left\| f - \sum_{k=-N}^N \lambda_k e_k \right\| < \epsilon$$

