The Bright Side of Mathematics

Fourier Transform – Part 9

$L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ has ONS $(\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots)$ given by $e_k : x \mapsto e^{ikx}$

$\rightsquigarrow$ Fourier series $\mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \langle e_k, f \rangle$

Parseval's identity: $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, f \rangle|^2$

$$\iff \|f - \mathcal{F}_n(f)\| \xrightarrow{n\to\infty} 0$$

means: $f = \mathcal{F}_n(f) + r_n$ with $\|r_n\| \xrightarrow{n\to\infty} 0$

Consider two functions: $f, g \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

$\langle f, g \rangle \leftarrow$ formula with Fourier coefficients?

$$f = \mathcal{F}_n(f) + r_n \quad \text{with} \quad \|r_n\| \xrightarrow{n\to\infty} 0$$

$$g = \mathcal{F}_n(g) + \tilde{r}_n \quad \text{with} \quad \|\tilde{r}_n\| \xrightarrow{n\to\infty} 0$$

We have: $|\langle \mathcal{F}_n(g), r_n \rangle| \underset{\text{Cauchy Schwarz}}{\leq} \|\mathcal{F}_n(g)\| \, \|r_n\|$

$\leq \|g\| \cdot \|r_n\| \xrightarrow{n\to\infty} 0$ (Bessel's inequality)

$$\langle f, g \rangle = \langle \mathcal{F}_n(f) + r_n, \mathcal{F}_n(g) + \tilde{r}_n \rangle$$

$$= \langle \mathcal{F}_n(f), \mathcal{F}_n(g) \rangle + \underbrace{\langle r_n, \mathcal{F}_n(g) \rangle + \langle \mathcal{F}_n(f), \tilde{r}_n \rangle + \langle r_n, \tilde{r}_n \rangle}_{(*)}$$

$$= \Big\langle \sum_{k=-n}^{n} e_k \langle e_k, f \rangle, \sum_{\ell=-n}^{n} e_\ell \langle e_\ell, g \rangle \Big\rangle + (*)$$

$$= \sum_{k=-n}^{n} \sum_{\ell=-n}^{n} \overline{\langle e_k, f \rangle} \langle e_\ell, g \rangle \underbrace{\langle e_k, e_\ell \rangle}_{= \delta_{k\ell}} + (*)$$

$$= \sum_{k=-n}^{n} \langle f, e_k \rangle \langle e_k, g \rangle + (*)$$

$$\xrightarrow{n\to\infty} \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle$$

Remember the equivalent statements: $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ with ONS $(e_k)_{k\in\mathbb{Z}}$

(a) Parseval's identity: $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, f \rangle|^2$

(b) ONS is complete: $\left\| f - \sum_{k=-n}^{n} e_k \langle e_k, f \rangle \right\| \xrightarrow{n\to\infty} 0$

$\left(f = \sum_{k=-\infty}^{\infty} e_k \langle e_k, f \rangle \right)$

(c) ONS gives inner product:

$$\langle f, g \rangle = \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle$$

informal: $\left(\sum_{k=-\infty}^{\infty} |e_k\rangle\langle e_k| = \mathbb{1} \right)$

(d) ONS is total: $\operatorname{Span}\left(\{e_k\}_{k\in\mathbb{Z}}\right)$ is dense in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$:

$$\forall f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \; \forall \varepsilon > 0 \; \exists N \in \mathbb{N}, \lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{C}:$$

$$\left\| f - \sum_{k=-N}^{N} \lambda_k e_k \right\| < \varepsilon$$