

## Fourier Transform - Part 8

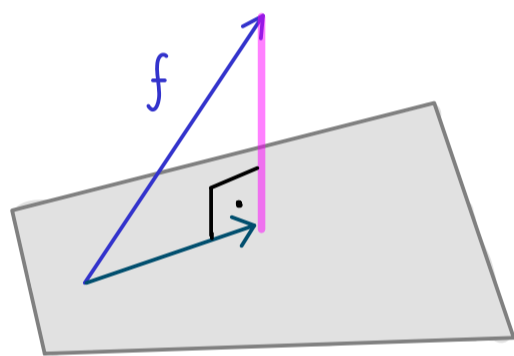
Fourier series:  $f \in L^1_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C}) \rightsquigarrow \mathcal{F}_n(f) \in \mathcal{P}_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$

trigonometric polynomial

$$\mathcal{F}_n(f) = \sum_{k=-n}^n c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$$

Geometric picture: For  $f \in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C}) \rightsquigarrow \mathcal{F}_n(f) \in \mathcal{P}_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$



orthogonal projection

$$\mathcal{F}_n(f) \perp \underbrace{f - \mathcal{F}_n(f)}_{\text{normal component}}$$

normal component

Question: What happens for  $n \rightarrow \infty$ ?  $\mathcal{F}_n(f) \xrightarrow{n \rightarrow \infty} f$ ?

Proposition:  $L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$  with inner product  $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \cdot g(x) dx$

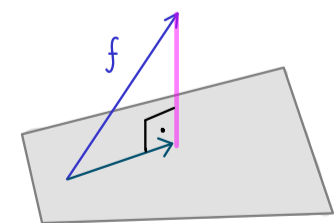
and ONS  $(\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots)$  given by  $e_k: x \mapsto e^{ikx}$ .

Then for  $f \in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$  and  $\mathcal{F}_n(f) = \sum_{k=-n}^n e_k \underbrace{\langle e_k, f \rangle}_{c_k}$ ,

we have:

$$(a) \quad \|f - \mathcal{F}_n(f)\|^2 = \|f\|^2 - \sum_{k=-n}^n |c_k|^2$$

$L^2$ -norm  $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$



Pythagorean theorem:  $\|f\|^2 = \|\mathcal{F}_n(f)\|^2 + \|f - \mathcal{F}_n(f)\|^2$

$$(b) \sum_{k=-n}^n |c_k|^2 \leq \|f\|^2 \quad \text{for all } n \quad (\text{Bessel's inequality})$$

$$\left( \Rightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 \leq \|f\|^2 \quad \text{and} \quad c_k \xrightarrow{k \rightarrow \infty} 0 \right)$$

$$(c) \|f - \mathcal{F}_n(f)\| \xrightarrow{n \rightarrow \infty} 0 \quad \Leftrightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 = \|f\|^2$$

(Parseval's identity)