

## Fourier Transform - Part 4

We already know: we have an orthogonal basis (OB)

$$\mathcal{B} = \left( \begin{array}{l} x \mapsto 1, \quad x \mapsto \cos(x), \quad x \mapsto \cos(2x), \quad x \mapsto \cos(3x), \dots, \\ x \mapsto \sin(x), \quad x \mapsto \sin(2x), \quad x \mapsto \sin(3x), \dots \end{array} \right)$$

for  $\mathcal{P}_{2\pi\text{-per}}$  with inner product  $\langle f, g \rangle_1 := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$

Normalize:

$$\langle x \mapsto \sin(kx), x \mapsto \sin(kx) \rangle_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sin(kx))^2 dx \quad \begin{array}{c} \uparrow \\ \square \\ \rightarrow \end{array}$$

$$\begin{aligned} \int_{-\pi}^{\pi} (\sin(kx))^2 dx &= \int_{-\pi}^{\pi} \underbrace{\sin(kx)}_u \underbrace{\sin(kx)}_{v'} dx = \sin(kx) \left(-\frac{1}{k}\right) \cos(kx) \Big|_{-\pi}^{\pi} \\ &\quad \text{integration by parts: } \begin{array}{l} u' = k \cos(kx) \\ v = -\frac{1}{k} \cos(kx) \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \\ &\quad - \int_{-\pi}^{\pi} k \cos(kx) \left(-\frac{1}{k}\right) \cos(kx) dx \\ &= \int_{-\pi}^{\pi} \underbrace{(\cos(kx))^2}_{1 - (\sin(kx))^2} dx \end{aligned}$$

$$\Rightarrow 2 \cdot \int_{-\pi}^{\pi} (\sin(kx))^2 dx = \int_{-\pi}^{\pi} 1 dx = 2\pi$$

$$\langle x \mapsto \sin(kx), x \mapsto \sin(kx) \rangle_1 = \frac{1}{2} \quad \rightsquigarrow \text{length} = \frac{1}{\sqrt{2}}$$

Hence:  $x \mapsto \sqrt{2} \cdot \sin(kx)$  has norm 1

Proposition: (1)  $\mathcal{B} = \left( \begin{array}{l} x \mapsto 1, \quad x \mapsto \sqrt{2} \cos(x), \quad x \mapsto \sqrt{2} \cos(2x), \quad x \mapsto \sqrt{2} \cos(3x), \dots, \\ x \mapsto \sqrt{2} \sin(x), \quad x \mapsto \sqrt{2} \sin(2x), \quad x \mapsto \sqrt{2} \sin(3x), \dots \end{array} \right)$

is an ONB w.r.t. the inner product:  $\langle f, g \rangle_1 := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$

$$(2) \quad \mathcal{B} = \left( x \mapsto \frac{1}{\sqrt{2\pi}}, x \mapsto \frac{1}{\sqrt{\pi}} \cos(x), x \mapsto \frac{1}{\sqrt{\pi}} \cos(2x), x \mapsto \frac{1}{\sqrt{\pi}} \cos(3x), \dots, \right. \\ \left. x \mapsto \frac{1}{\sqrt{\pi}} \sin(x), x \mapsto \frac{1}{\sqrt{\pi}} \sin(2x), x \mapsto \frac{1}{\sqrt{\pi}} \sin(3x), \dots \right)$$

is an ONB w.r.t. the inner product:  $\langle f, g \rangle_2 := \int_{-\pi}^{\pi} f(x)g(x) dx$

$$(3) \quad \mathcal{B} = \left( x \mapsto \frac{1}{\sqrt{2}}, x \mapsto \cos(x), x \mapsto \cos(2x), x \mapsto \cos(3x), \dots, \right. \\ \left. x \mapsto \sin(x), x \mapsto \sin(2x), x \mapsto \sin(3x), \dots \right)$$

is an ONB w.r.t. the inner product:  $\langle f, g \rangle_3 := \frac{1}{\sqrt{2}} \int_{-\pi}^{\pi} f(x)g(x) dx$

For trigonometric polynomials:

$$f(x) = \tilde{a}_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^n a_k \cos(k \cdot x) + \sum_{k=1}^n b_k \sin(k \cdot x), \quad a_i, b_i \in \mathbb{R}$$

Fourier coefficients w.r.t. ONB in (3)

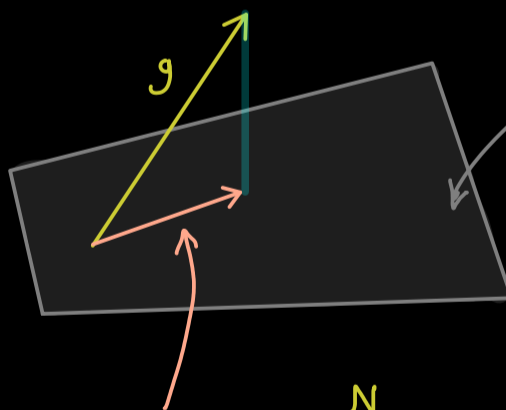
$$a_k = \langle x \mapsto \cos(k \cdot x), f \rangle_3, \quad \tilde{a}_0 = \langle x \mapsto \frac{1}{\sqrt{2}}, f \rangle_3$$

$$b_k = \langle x \mapsto \sin(k \cdot x), f \rangle_3$$

Approximation of periodic functions?

$$g : \mathbb{R} \rightarrow \mathbb{R}$$

$2\pi$ -periodic + "integrable"



trigonometric polynomials with basis:

$$\mathcal{B} = (h_1, h_2, \dots, h_N)$$

ONB!

$$\text{orthogonal projection} = \sum_{k=1}^N h_k \langle h_k, g \rangle$$