The Bright Side of
Mathematics

Fourier Transform – Part 4

We already know: we have an orthogonal basis (OB)

$$\mathcal{B} = \big(x \mapsto 1,\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x),\ \dots,\ x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x),\ \dots\big)$$

for $\mathcal{P}_{2\pi\text{-per}}$ with inner product $\langle f, g \rangle_1 := \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

Normalize: $\langle x \mapsto \sin(kx),\ x \mapsto \sin(kx) \rangle_1 = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} (\sin(kx))^2\, dx$

$$\int_{-\pi}^{\pi} (\sin(kx))^2\, dx = \int_{-\pi}^{\pi} \underbrace{\sin(kx)}_{u}\ \underbrace{\sin(kx)}_{v'}\, dx = \sin(kx)\left(-\tfrac{1}{k}\right)\cos(kx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} k\cos(kx)\left(-\tfrac{1}{k}\right)\cos(kx)\, dx$$

integration by parts: $u' = k\cos(kx)$, $v = -\frac{1}{k}\cos(kx)$

$$= \int_{-\pi}^{\pi} \underbrace{(\cos(kx))^2}_{1-(\sin(kx))^2}\, dx$$

$$\Rightarrow\ 2 \cdot \int_{-\pi}^{\pi} (\sin(kx))^2\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi$$

$$\langle x \mapsto \sin(kx),\ x \mapsto \sin(kx) \rangle_1 = \frac{1}{2} \quad \rightsquigarrow \quad \text{length} = \frac{1}{\sqrt{2}}$$

Hence: $x \mapsto \sqrt{2} \cdot \sin(kx)$ has norm 1

Proposition:

(1) $$\mathcal{B} = \big(x \mapsto 1,\ x \mapsto \sqrt{2}\cos(x),\ x \mapsto \sqrt{2}\cos(2x),\ x \mapsto \sqrt{2}\cos(3x),\ \dots,\ x \mapsto \sqrt{2}\sin(x),\ x \mapsto \sqrt{2}\sin(2x),\ x \mapsto \sqrt{2}\sin(3x),\ \dots\big)$$

is an ONB w.r.t. the inner product: $\langle f, g \rangle_1 := \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

(2) $$\mathcal{B} = \big(x \mapsto \tfrac{1}{\sqrt{2\pi}},\ x \mapsto \tfrac{1}{\sqrt{\pi}}\cos(x),\ x \mapsto \tfrac{1}{\sqrt{\pi}}\cos(2x),\ x \mapsto \tfrac{1}{\sqrt{\pi}}\cos(3x),\ \dots,\ x \mapsto \tfrac{1}{\sqrt{\pi}}\sin(x),\ x \mapsto \tfrac{1}{\sqrt{\pi}}\sin(2x),\ x \mapsto \tfrac{1}{\sqrt{\pi}}\sin(3x),\ \dots\big)$$

is an ONB w.r.t. the inner product: $\langle f, g \rangle_2 := \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

(3) $$\mathcal{B} = \big(x \mapsto \tfrac{1}{\sqrt{2}},\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x),\ \dots,\ x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x),\ \dots\big)$$

is an ONB w.r.t. the inner product: $\langle f, g \rangle_3 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

For trigonometric polynomials:

$$f(x) = \tilde{a}_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x), \quad a_i, b_i \in \mathbb{R}$$

Fourier coefficients w.r.t. ONB in (3)

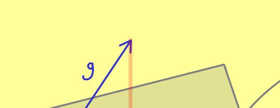
$$a_k = \langle x \mapsto \cos(k \cdot x),\ f \rangle_3, \qquad \tilde{a}_0 = \langle x \mapsto \tfrac{1}{\sqrt{2}},\ f \rangle_3$$

$$b_k = \langle x \mapsto \sin(k \cdot x),\ f \rangle_3$$

Approximation of periodic functions?

$g : \mathbb{R} \to \mathbb{R}$

2π-periodic + "integrable"



trigonometric polynomials with basis:

$\mathcal{B} = (h_1, h_2, \dots, h_N)$

ONB!

orthogonal projection $= \sum_{k=1}^{N} h_k \langle h_k, g \rangle$