Fourier Transform – Part 3

In $\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{R})$, we have (real) trigonometric polynomials:

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x) \quad , \quad a_i, b_i \in \mathbb{R}$$

Subspace: $\mathcal{P}_{2\pi\text{-per}} := \mathrm{Span}\big(\underbrace{x \mapsto 1,\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x), \dots,\ x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \dots}_{\text{basis!}} \big)$

<u>Definition:</u> For $f, g \in \mathcal{P}_{2\pi\text{-per}}$, we define an inner product:

$$\langle f, g \rangle := \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

<u>Example:</u>

$$\langle x \mapsto 1, x \mapsto 1 \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} 1 \, dx = 1$$

$$\langle x \mapsto \cos(x), x \mapsto \sin(x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(x) \sin(x)\, dx$$

$$= \frac{1}{2\pi} \left(\frac{1}{2} (\sin(x))^2 \Big|_{-\pi}^{\pi} \right) = 0$$

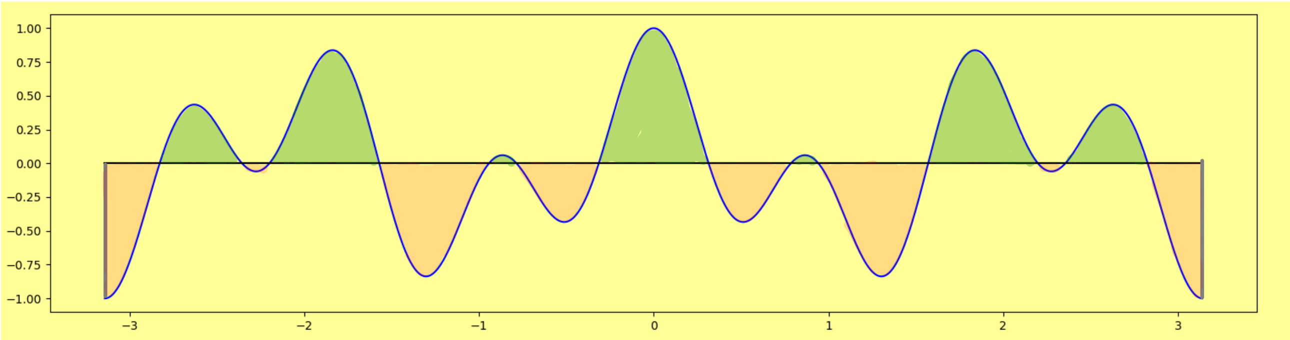
$$\langle x \mapsto \cos(k \cdot x), x \mapsto \sin(m \cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \underbrace{\cos(k \cdot x) \sin(m \cdot x)}_{\text{odd function}} \, dx = 0$$

$$\langle x \mapsto 1, x \mapsto \cos(k \cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(k \cdot x)\, dx = \frac{1}{2\pi} \frac{1}{k} \sin(k \cdot x) \Big|_{-\pi}^{\pi} = 0$$

$$\langle x \mapsto 1, x \mapsto \sin(m \cdot x) \rangle = 0$$

$$\left\langle x \mapsto \cos(k\cdot x)\,,\; x \mapsto \cos(m\cdot x)\right\rangle = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} \cos(k\cdot x)\,\cos(m\cdot x)\,dx$$

$$= 0 \qquad \text{if } k \neq m$$



Use: $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

Then: $$\int_{-\pi}^{\pi} \cos(k\cdot x)\,\cos(m\cdot x)\,dx = \frac{1}{4}\int_{-\pi}^{\pi}\Big(e^{i(k+m)x} + e^{-i(k+m)x} + e^{i(k-m)x} + e^{-i(k-m)x}\Big)dx$$

$k \neq m$

$$= \frac{1}{4}\left(\frac{1}{i(k+m)}e^{i(k+m)x} + \frac{1}{-i(k+m)}e^{-i(k+m)x} + \frac{1}{i(k-m)}e^{i(k-m)x} + \frac{1}{-i(k-m)}e^{-i(k-m)x}\right)\Bigg|_{-\pi}^{\pi}$$

Use: $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

$$= \frac{1}{2}\left(\frac{1}{k+m}\sin((k+m)\cdot x) + \frac{1}{k+m}\sin((k-m)\cdot x)\right)\Bigg|_{-\pi}^{\pi} = 0$$

And similarly: $$\int_{-\pi}^{\pi} \sin(k\cdot x)\,\sin(m\cdot x)\,dx \overset{k\neq m}{=} 0$$

<u>Result:</u> $$\mathcal{B} = \Big(x \mapsto 1\,,\; x\mapsto \cos(x)\,,\; x \mapsto \cos(2x)\,,\; x\mapsto \cos(3x)\,,\ldots,$$
$$x\mapsto \sin(x)\,,\; x\mapsto \sin(2x)\,,\; x\mapsto\sin(3x)\,,\ldots \Big)$$

satisfies $\langle f, g\rangle = 0 \quad f \neq g\,,\; f,g \in \mathcal{B}$

$\rightsquigarrow$ $\mathcal{B}$ orthogonal basis (OB)

$\rightsquigarrow$ make to orthonormal basis (ONB)