Fourier Transform — Part 3

In $\mathcal{F}_{2\pi\text{-per}}(\mathbb{R}, \mathbb{R})$, we have (real) trigonometric polynomials:

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x) \quad , \quad a_i, b_i \in \mathbb{R}$$

Subspace: $P_{2\pi\text{-per}} := \text{span}\Big(x \mapsto 1 \, , \; x \mapsto \cos(x), \; x \mapsto \cos(2x), \; x \mapsto \cos(3x) \, , ..., $

$\qquad\qquad\qquad x \mapsto \sin(x), \; x \mapsto \sin(2x), \; x \mapsto \sin(3x) \, , ... \Big)$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{basis!}}$

Definition: For $f, g \in P_{2\pi\text{-per}}$, we define an inner product:

$$\langle f, g \rangle := \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x) \, g(x) \, dx$$

Example:

$$\langle x \mapsto 1, \; x \mapsto 1 \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} 1 \, dx = 1$$

$$\langle x \mapsto \cos(x), \; x \mapsto \sin(x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(x) \sin(x) \, dx$$

$$= \frac{1}{2\pi} \left(\frac{1}{2} \big(\sin(x)\big)^2 \Big|_{-\pi}^{\pi} \right) = 0$$

$$\langle x \mapsto \cos(k \cdot x), \; x \mapsto \sin(m \cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \underbrace{\cos(k \cdot x) \sin(m \cdot x)}_{\text{odd function}} \, dx = 0$$
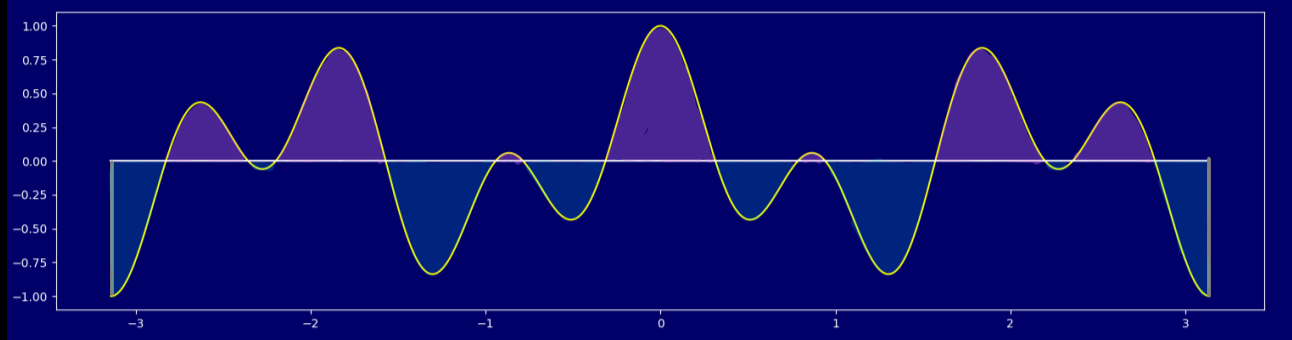
$$\langle x \mapsto 1, \; x \mapsto \cos(k \cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(k \cdot x) \, dx = \frac{1}{2\pi} \, \frac{1}{k} \sin(k \cdot x) \Big|_{-\pi}^{\pi} = 0$$

$$\langle x \mapsto 1, \; x \mapsto \sin(m \cdot x) \rangle = 0$$

$$\left\langle x \mapsto \cos(k \cdot x) , x \mapsto \cos(m \cdot x) \right\rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(k \cdot x) \cos(m \cdot x) \, dx$$

$$= 0 \qquad \text{if } k \neq m$$

Use: $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

Then: $\int_{-\pi}^{\pi} \cos(k \cdot x) \cos(m \cdot x) \, dx = \frac{1}{4} \int_{-\pi}^{\pi} \left(e^{i(k+m)x} + e^{-i(k+m)x} \right.$

$$\left. + e^{i(k-m)x} + e^{-i(k-m)x} \right) dx$$

$k \neq m$

$$= \frac{1}{4}\left(\frac{1}{i(k+m)} e^{i(k+m)x} + \frac{1}{-i(k+m)} e^{-i(k+m)x} + \frac{1}{i(k-m)} e^{i(k-m)x} + \frac{1}{-i(k-m)} e^{-i(k-m)x} \right)\bigg|_{-\pi}^{\pi}$$

Use: $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

$$= \frac{1}{2}\left(\frac{1}{k+m} \sin((k+m)\cdot x) + \frac{1}{k+m} \sin((k-m)\cdot x) \right)\bigg|_{-\pi}^{\pi} = 0$$

$k \neq m$

And similarly: $\int_{-\pi}^{\pi} \sin(k \cdot x) \sin(m \cdot x) \, dx = 0$

Result: $\mathcal{B} = \Big(x \mapsto 1 , x \mapsto \cos(x), x \mapsto \cos(2x), x \mapsto \cos(3x), \ldots,$

$$x \mapsto \sin(x), x \mapsto \sin(2x), x \mapsto \sin(3x), \ldots \Big)$$

satisfies $\left\langle f , g \right\rangle = 0 \qquad f \neq g , \quad f, g \in \mathcal{B}$

$\rightsquigarrow \mathcal{B}$ orthogonal basis (OB)

$\longrightarrow$ make to orthonormal basis (ONB)