BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Fourier Transform – Part 3

In $\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{R})$, we have (real) trigonometric polynomials:

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cdot \cos(k\cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x) \quad , \quad a_i, b_i \in \mathbb{R}$$

Subspace: $\mathcal{P}_{2\pi\text{-per}} := \operatorname{Span}\big(x \mapsto 1,\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x), \ldots,$
$x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \ldots \big)$

basis!

Definition: For $f, g \in \mathcal{P}_{2\pi\text{-per}}$, we define an inner product:

$$\langle f, g \rangle := \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

Example:

$$\langle x \mapsto 1, x \mapsto 1 \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} 1\, dx = 1$$

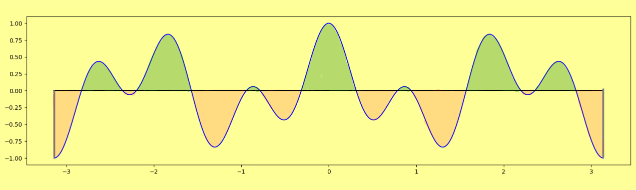
$$\langle x \mapsto \cos(x), x \mapsto \sin(x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(x) \sin(x)\, dx$$
$$= \frac{1}{2\pi} \left(\frac{1}{2} (\sin(x))^2 \Big|_{-\pi}^{\pi} \right) = 0$$

$$\langle x \mapsto \cos(k\cdot x), x \mapsto \sin(m \cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \underbrace{\cos(k\cdot x) \sin(m\cdot x)}_{\text{odd function}}\, dx = 0$$

$$\langle x \mapsto 1, x \mapsto \cos(k\cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(k\cdot x)\, dx = \frac{1}{2\pi} \frac{1}{k} \sin(k\cdot x) \Big|_{-\pi}^{\pi} = 0$$

$$\langle x \mapsto 1, x \mapsto \sin(m\cdot x) \rangle = 0$$

$$\langle x \mapsto \cos(k\cdot x), x \mapsto \cos(m\cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(k\cdot x) \cos(m\cdot x)\, dx$$



$$= 0 \quad \text{if } k \neq m$$

Use: $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

Then: $$\int_{-\pi}^{\pi} \cos(k\cdot x) \cos(m\cdot x)\, dx = \frac{1}{4} \int_{-\pi}^{\pi} \Big(e^{i(k+m)x} + e^{-i(k+m)x} + e^{i(k-m)x} + e^{-i(k-m)x} \Big) dx$$

$$\overset{k \neq m}{=} \frac{1}{4} \left(\frac{1}{i(k+m)} e^{i(k+m)x} + \frac{1}{-i(k+m)} e^{-i(k+m)x} + \frac{1}{i(k-m)} e^{i(k-m)x} + \frac{1}{-i(k-m)} e^{-i(k-m)x} \right) \Big|_{-\pi}^{\pi}$$

Use: $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

$$= \frac{1}{2} \left(\frac{1}{k+m} \sin((k+m)\cdot x) + \frac{1}{k+m} \sin((k-m)\cdot x) \right) \Big|_{-\pi}^{\pi} = 0$$

And similarly: $$\int_{-\pi}^{\pi} \sin(k\cdot x) \sin(m\cdot x)\, dx \overset{k \neq m}{=} 0$$

Result: $\mathcal{B} = \big(x \mapsto 1,\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x), \ldots,$
$x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \ldots \big)$

satisfies $\langle f, g \rangle = 0 \quad f \neq g,\ f, g \in \mathcal{B}$

$\rightsquigarrow$ $\mathcal{B}$ orthogonal basis (OB)

$\rightsquigarrow$ make to orthonormal basis (ONB)