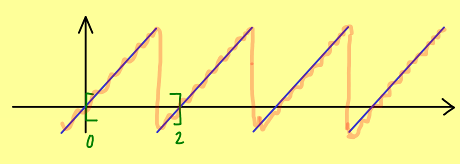
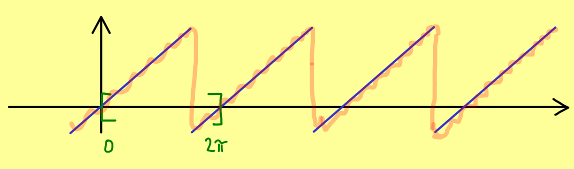
The Bright Side of Mathematics

Fourier Transform – Part 2

Idea of Fourier series:



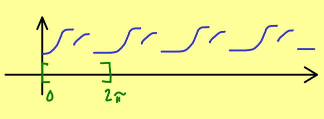
The function is 2-periodic: $f(x+2) = f(x)$ for all $x \in \mathbb{R}$

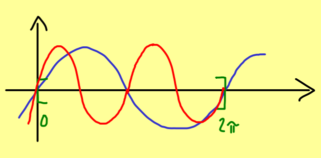


$$\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R} \}$$

↳ real vector space

Example:

- constant function $f(x) = 5$
- 

- 

$x \mapsto \sin(x)$
$x \mapsto \sin(2x)$

Proposition: $U \subseteq \mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{R})$ given by

$$U := \left\{ \begin{matrix} x \mapsto \sin(x), & x \mapsto \sin(2x), & x \mapsto \sin(3x), & \dots, \\ x \mapsto 1, & x \mapsto \cos(x), & x \mapsto \cos(2x), & x \mapsto \cos(3x), \dots \end{matrix} \right\}$$

(odd functions: first row; even functions: second row)

is linearly independent.

Definition: A linear combination $f \in \mathrm{Span}(U)$, $f : \mathbb{R} \to \mathbb{R}$, is called (real) trigonometric polynomial:

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x) \quad , \quad a_i, b_i \in \mathbb{R}$$

For $\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$, we have a (complex) trigonometric polynomial:

$$f(x) = \sum_{k=-n}^{n} c_k \cdot \exp(i \cdot k \cdot x) \quad , \quad c_k \in \mathbb{C}$$