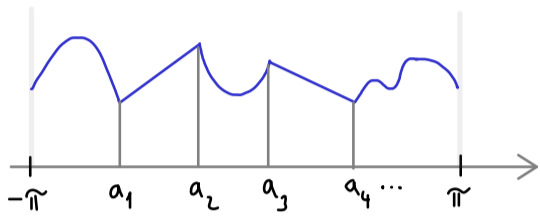# Fourier Transform - Part 15

Theorem: $f: \mathbb{R} \longrightarrow \mathbb{C}$ 2π-periodic <u>continuous</u> function
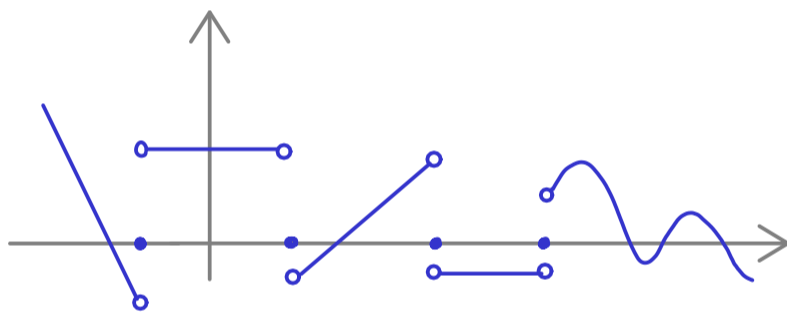
and piecewise C^1-function:

there are finitely many points $(a_1, a_2, \ldots, a_m)$

inside the interval $[-\pi, \pi]$ such that: $f|_{[a_j, a_{j+1}]} \in C^1$

for all $j \in \{0, 1, \ldots, m\}$, $a_0 := -\pi$, $a_{m+1} := \pi$

Then: $\mathcal{F}_n(f) \xrightarrow{\ n \to \infty\ } f$ <u>uniformly.</u>

Proof: Consider the derivative function: $\tilde{f}(x) := \begin{cases} 0 & , \quad x \in \{a_0, a_1, \ldots, a_{m+1}\} \\ f'(x) & , \quad \text{else} \end{cases}$

piecewise continuous function
$\in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$

Parseval's identity: $\|\tilde{f}\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, \tilde{f}\rangle|^2 < \infty$

What about the Fourier coefficients of f? $(k \neq 0)$

$$c_k := \langle e_k, f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{e^{-ikx}}_{u'} \underbrace{f(x)}_{v}\, dx = \frac{1}{2\pi}\left(u \cdot v \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u\, v'\, dx \right)$$

$u = \frac{1}{-ik} e^{-ikx}$ integration by parts

$$= \frac{1}{2\pi}\left(0 + \frac{1}{ik}\int_{-\pi}^{\pi} e^{-ikx}\, \tilde{f}(x)\, dx \right) = \frac{1}{ik}\langle e_k, \tilde{f}\rangle$$

> **General inequality for real numbers:** $x \cdot y \leq \dfrac{x^2 + y^2}{2}$

$$|c_k| = \frac{1}{k}\left|\langle e_k, \tilde{f}\rangle\right| \leq \frac{1}{2}\left(\frac{1}{k^2} + \left|\langle e_k, \tilde{f}\rangle\right|^2 \right)$$

$$\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} |C_k| \leq \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k^2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} |\langle e_k, \tilde{f} \rangle|^2 < \infty$$

$$\mathcal{F}_n(f)(x) = \sum_{k=-n}^{n} \underbrace{e^{ikx} \cdot C_k}_{f_k(x)} \qquad \text{with} \quad |f_k(x)| \leq M_k =: |C_k|, \quad \sum_{k=-\infty}^{\infty} M_k < \infty$$

Weierstrass
M-Test

$$\Longrightarrow \quad \sum_{k=-\infty}^{\infty} f_k \qquad \text{uniformly convergent to a continuous function}$$

$$h : [-\pi, \pi] \longrightarrow \mathbb{C}$$

Status quo: $\quad \left\| \mathcal{F}_n(f) - h \right\|_\infty \xrightarrow{h \to \infty} 0 \ , \quad \left\| \mathcal{F}_n(f) - f \right\|_{L^2} \xrightarrow{h \to \infty} 0$

More estimates: $\quad \left\| f - h \right\|_{L^2} \leq \left\| f - \mathcal{F}_n(f) \right\|_{L^2} + \underbrace{\left\| \mathcal{F}_n(f) - h \right\|_{L^2}}$

$$\leq \left\| \mathcal{F}_n(f) - h \right\|_\infty$$

$$\xrightarrow{h \to \infty} 0 \qquad \text{\textcolor{blue}{continuous functions}}$$

Hence: $\quad \left\| f - h \right\|_{L^2} = 0 \quad \Longrightarrow \quad f = h$

Conclusion: $\quad \left\| \mathcal{F}_n(f) - f \right\|_\infty \xrightarrow{h \to \infty} 0 \qquad$ (uniform convergence of the Fourier series)

$\square$