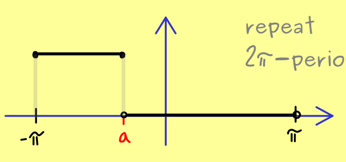
The Bright Side of Mathematics

Fourier Transform – Part 12

h_a:

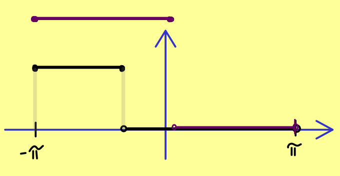


$$h_a(x) = \begin{cases} 1, & x \in [-\pi, a] \\ 0, & x \in (a, \pi) \end{cases}$$

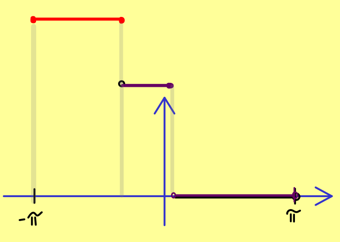
Parseval's identity holds for h_a for every possible a. (part 10)

Step functions: consider the complex vector space:

$$\mathcal{S}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) := \left\{ g \in \mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \;\middle|\; \text{there are } m \in \mathbb{N},\ a_i \in [-\pi,\pi],\ \lambda_i \in \mathbb{C} \text{ such that: } g = \sum_{i=1}^{m} \lambda_i \cdot h_{a_i} \right\}$$



adding



Do we have Parseval's identity here?

Consider step function $g \in \mathcal{S}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

$$g = \sum_{i=1}^{m} \lambda_i \cdot h_{a_i}$$

$$c_k = \langle e_k, g \rangle = \left\langle e_k, \sum_{i=1}^{m} \lambda_i \cdot h_{a_i} \right\rangle = \sum_{i=1}^{m} \lambda_i \langle e_k, h_{a_i} \rangle$$

$$|c_k|^2 = \overline{c_k}\, c_k = \overline{\sum_{j=1}^{m} \lambda_j \langle e_k, h_{a_j} \rangle} \cdot \sum_{i=1}^{m} \lambda_i \langle e_k, h_{a_i} \rangle$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{m} \overline{\lambda_j} \lambda_i \langle h_{a_j}, e_k \rangle \langle e_k, h_{a_i} \rangle$$

$$\sum_{k=-n}^{n} |c_k|^2 = \sum_{i,j=1}^{m} \overline{\lambda_j} \lambda_i \left(\sum_{k=-n}^{n} \langle h_{a_j}, e_k \rangle \langle e_k, h_{a_i} \rangle \right) \xrightarrow{n \to \infty} \langle h_{a_j}, h_{a_i} \rangle$$

(part 9)

informal: $\left(\sum_{k=-\infty}^{\infty} |e_k\rangle\langle e_k| = \mathbb{1} \right)$

we have Parseval's identity for h_{a_j} and h_{a_i}

$$\Longrightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 = \sum_{i,j=1}^{m} \overline{\lambda_j} \lambda_i \langle h_{a_j}, h_{a_i} \rangle = \left\langle \sum_{j=1}^{m} \lambda_j \cdot h_{a_j}, \sum_{i=1}^{m} \lambda_i \cdot h_{a_i} \right\rangle$$

$$= \langle g, g \rangle = \|g\|^2$$

Result: Parseval's identity holds for $\mathcal{S}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \subseteq L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$.