The Bright Side of Mathematics

Fourier Transform – Part 8

Fourier series: $f \in L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \rightsquigarrow \mathcal{F}_n(f) \in \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

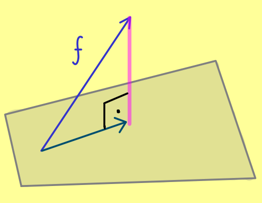
trigonometric polynomial

$$\mathcal{F}_n(f) = \sum_{k=-n}^{n} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$$

Geometric picture: For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \rightsquigarrow \mathcal{F}_n(f) \in \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

orthogonal projection



$\mathcal{F}_n(f) \perp \underbrace{f - \mathcal{F}_n(f)}$ normal component

Question: What happens for $n \to \infty$? $\mathcal{F}_n(f) \xrightarrow{n\to\infty} f$?

Proposition: $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ with inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\cdot g(x)\, dx$

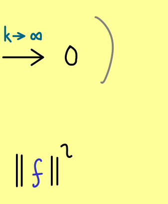
and ONS $(\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots)$ given by $e_k : x \mapsto e^{ikx}$.

Then for $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ and $\mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \underbrace{\langle e_k, f\rangle}_{c_k}$,

we have:

(a) $\| f - \mathcal{F}_n(f)\|^2 = \|f\|^2 - \sum_{k=-n}^{n} |c_k|^2$

L^2-norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$



Pythagorean theorem: $\|f\|^2 = \|\mathcal{F}_n(f)\|^2 + \|f - \mathcal{F}_n(f)\|^2$

(b) $\sum_{k=-n}^{n} |c_k|^2 \le \|f\|^2$ for all n (Bessel's inequality)

$\left(\Rightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 \le \|f\|^2 \quad \text{and} \quad c_k \xrightarrow{k\to\infty} 0 \right)$

(c) $\|f - \mathcal{F}_n(f)\| \xrightarrow{n\to\infty} 0 \iff \sum_{k=-\infty}^{\infty} |c_k|^2 = \|f\|^2$

(Parseval's identity)