Ordinary Differential Equations
Exercises 1

Exercise 1. Find the solutions of the following ODEs:

a) $x^2 y' = y^2$

b) $y'(1+x^2) = xy$

c) $y' = (1-y^2)$

d) $y' \sin y = -x$

a) $x^2 y' = y^2$

$y' = f(x)g(y)$ separation of variables

$y' = y^2 \frac{1}{x^2}$ $\frac{dy}{dx}\frac{1}{y^2} = \frac{1}{x^2}$, $\int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx$, $-\frac{1}{y} = -\frac{1}{x} + c$

$$\boxed{y = \frac{x}{1-cx}, \quad c \in \mathbb{R}}$$

b) $y'(1+x^2) = xy$

$y' = \frac{x}{(1+x^2)} y$

$\frac{dy}{dx}\frac{1}{y} = \frac{x}{1+x^2}$

$\int \frac{1}{y} dy = \frac{1}{2}\int \frac{2x}{1+x^2} dx$ $u = 1+x^2$, $\frac{du}{dx} = 2x$

$\ln|y| = \frac{1}{2}\int \frac{1}{u} du$

$\ln|y| = \frac{1}{2}\ln(1+x^2) + c$

$e^{\ln|y|} = e^{\ln(1+x^2)^{1/2} + c}$

$\pm y = (1+x^2)^{1/2} \cdot e^c$

$y = \sqrt{1+x^2} \cdot \underbrace{(\pm e^c)}_{\hat{c}}$

$$\boxed{y = \sqrt{1+x^2}\,\hat{c}, \quad \hat{c} \in \mathbb{R}}$$

c) $y' = 1 - y^2$

$\frac{d}{dx} \operatorname{artanh}(x) = \frac{1}{1-x^2}, \quad |x| < 1$

$\frac{d}{dx} \operatorname{arcotanh}(x) = \frac{1}{1-x^2}, \quad |x| > 1$

$y' = f(x)\, g(y)$

$\frac{dy}{dx} \frac{1}{1-y^2} = 1$

$\int \frac{1}{1-y^2}\, dy = \int 1\, dx$

$\ln \frac{|y+1|}{|y-1|} = (x + C)2$

$\frac{|y+1|}{|y-1|} = e^{2x + C}$

$\frac{1}{1-y^2} = \frac{A}{(y-1)} + \frac{B}{(y+1)}$

$\times(y-1) \quad \frac{\cancel{(y-1)}}{-\cancel{(1-y)}(1+y)} = A + \frac{(y-1)B}{(y+1)} \xrightarrow{y=1} -\frac{1}{2} = A$

$\times(y+1) \quad \frac{1}{1-y} = -\frac{1}{2}\frac{1}{(y-1)}(y+1) + B \xrightarrow{y=-1} \frac{1}{2} = B$

$\int \frac{1}{1-y^2}\, dy = -\frac{1}{2}\int \frac{1}{y-1}\, dy + \frac{1}{2}\int \frac{1}{y+1}\, dy = -\frac{1}{2}\ln|y-1| + \frac{1}{2}\ln|y+1| = \frac{1}{2}\ln\frac{|y+1|}{|y-1|}$

① $|y| > 1, \quad \frac{y+1}{y-1} = e^{2x+C}$

② $|y| < 1, \quad \frac{y+1}{1-y} = e^{2x+C}$

$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$

① $y + 1 = y e^{2x+C} - e^{2x+C}$

$y(1 - e^{2x+C}) = -1 - e^{2x+C}$

$y = \frac{1 + e^{2x+C}}{e^{2x+C} - 1} \quad \frac{e^{-x-\frac{C}{2}}}{e^{-x-\frac{C}{2}}}$

$y = \frac{e^{-(x+\frac{C}{2})} + e^{x+\frac{C}{2}}}{e^{x+\frac{C}{2}} - e^{-(x+\frac{C}{2})}}$

$\boxed{y = \operatorname{cotanh}(x + \hat{C}), \quad |y| > 1, \quad \hat{C} \in \mathbb{R}}$

② $y + 1 = e^{2x+C} - y e^{2x+C}$

$y(1 + e^{2x+C}) = e^{2x+C} - 1$

$y = \frac{e^{x+\frac{C}{2}} - e^{-(x+\frac{C}{2})}}{e^{-(x+\frac{C}{2})} + e^{x+\frac{C}{2}}}$

$\boxed{y = \tanh(x + \hat{C}), \quad |y| < 1, \quad \hat{C} \in \mathbb{R}}$

d) $y' \sin y = -x$

$y' = -x \cdot \frac{1}{\sin y}$

$\frac{dy}{dx} \sin y = -x$

$\int \sin y\, dy = -\int x\, dx$

$-\cos y = -\frac{x^2}{2} + C$

$\boxed{y = \arccos\left(\frac{x^2}{2} - C\right), \quad C \in \mathbb{R}}$