



# Ordinary Differential Equations

## Exercises 1

Exercise 1. Find the solutions of the following ODEs:

a)  $x^2 y' = y^2$

c)  $y' = (1 - y^2)$

b)  $y'(1 + x^2) = xy$

d)  $y' \sin y = -x$

a)  $x^2 y' = y^2$

$y' = f(x)g(y)$  separation of variables

$$y' = y^2 \frac{1}{x^2} \quad \frac{dy}{dx} \frac{1}{y^2} = \frac{1}{x^2}, \quad \int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx, \quad -\frac{1}{y} = -\frac{1}{x} + c$$

$$y = \frac{x}{1 - cx}, \quad c \in \mathbb{R}$$

b)  $y'(1 + x^2) = xy$

$$y' = \frac{x}{(1+x^2)} y$$

$$\ln|y| = \frac{1}{2} \ln(1+x^2) + c$$

$$\frac{dy}{dx} \frac{1}{y} = \frac{x}{1+x^2}$$

$$e^{\ln|y|} = e^{\ln(1+x^2)^{1/2} + c}$$

$$\pm y = (1+x^2)^{1/2} \cdot e^c$$

$$\int \frac{1}{y} dy = \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$u = 1+x^2 \\ \frac{du}{dx} = 2x$$

$$y = \sqrt{1+x^2} \cdot \underbrace{(\pm e^c)}_{\hat{c}}$$

$$\ln|y| = \frac{1}{2} \int \frac{1}{u} du$$

$$y = \sqrt{1+x^2} \hat{c}, \quad \hat{c} \in \mathbb{R}$$

$$c) y' = 1 - y^2$$

$$y' = f(x)g(y)$$

$$\frac{1}{1-y^2} = \frac{A}{(y-1)} + \frac{B}{(y+1)}$$

$$\frac{dy}{dx} \frac{1}{1-y^2} = 1$$

$$x(y-1) \frac{(y+1)}{-(1-y)(1+y)} = A + \frac{(y-1)B}{(y+1)} \xrightarrow{y=1} -\frac{1}{2} = A$$

$$\int \frac{1}{1-y^2} dy = \int 1 dx$$

$$\ln \frac{|y+1|}{|y-1|} = (x+c)2$$

$$x(y+1) \frac{1}{1-y} = -\frac{1}{2} \frac{1}{(y-1)}(y+1) + B \xrightarrow{y=-1} \frac{1}{2} = B$$

$$\frac{|y+1|}{|y-1|} = e^{2x+c}$$

$$\int \frac{1}{1-y^2} dy = -\frac{1}{2} \int \frac{1}{y-1} dy + \frac{1}{2} \int \frac{1}{y+1} dy = -\frac{1}{2} \ln|y-1| + \frac{1}{2} \ln|y+1| = \frac{1}{2} \ln \frac{|y+1|}{|y-1|}$$

$$① |y| > 1, \frac{y+1}{y-1} = e^{2x+c}$$

$$② |y| < 1, \frac{y+1}{1-y} = e^{2x+c}$$

$$① \quad y+1 = y e^{2x+c} - e^{2x+c}$$

$$y(1 - e^{2x+c}) = -1 - e^{2x+c}$$

$$y = \frac{1 + e^{2x+c}}{e^{2x+c} - 1} = \frac{e^{-x-\frac{c}{2}} + e^{x+\frac{c}{2}}}{e^{x+\frac{c}{2}} - e^{-x-\frac{c}{2}}}$$

$$y = \frac{e^{-(x+\frac{c}{2})} + e^{x+\frac{c}{2}}}{e^{x+\frac{c}{2}} - e^{-x-\frac{c}{2}}}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$y = \coth \left( x + \hat{c} \right), \quad |y| > 1, \quad \hat{c} \in \mathbb{R}$$

$$② \quad y+1 = e^{2x+c} - y e^{2x+c}$$

$$y(1 + e^{2x+c}) = e^{2x+c} - 1$$

$$y = \frac{e^{x+\frac{c}{2}} - e^{-(x+\frac{c}{2})}}{e^{-(x+\frac{c}{2})} + e^{x+\frac{c}{2}}}$$

$$y = \tanh \left( x + \hat{c} \right), \quad |y| < 1, \quad \hat{c} \in \mathbb{R}$$

$$d) y' \sin y = -x$$

$$y' = -x \cdot \frac{1}{\sin y}$$

$$\frac{dy}{dx} \sin y = -x$$

$$\int \sin y dy = - \int x dx$$

$$-\cos y = -\frac{x^2}{2} + C$$

$$y = \arccos \left( \frac{x^2}{2} - C \right), \quad C \in \mathbb{R}$$