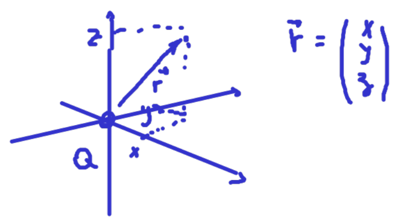
Multivariable Calculus

Exercises 1

Exercise 1. The electrostatic potential of a charged particle Q sitting in the origin of a three-dimensional coordinate system at a point $\vec{r}$ is given by

$$U(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\|\vec{r}\|}$$



where $\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$.

Compute the electrical field $\vec{E}(\vec{r})$, which is given by the negative gradient of the potential

$$\vec{E}(\vec{r}) = -\vec{\nabla} U(\vec{r}).$$

$$\vec{\nabla} U(\vec{r}) = \begin{pmatrix} \frac{\partial U}{\partial x}(\vec{r}) \\ \frac{\partial U}{\partial y}(\vec{r}) \\ \frac{\partial U}{\partial z}(\vec{r}) \end{pmatrix}$$

$$\frac{\partial U}{\partial x}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{\partial}{\partial x} (x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{Q}{4\pi\varepsilon_0} \left(-\frac{1}{2}\right) (x^2+y^2+z^2)^{-\frac{3}{2}} (2x) = \frac{Q}{4\pi\varepsilon_0} (-x) \frac{1}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$= \frac{Q}{4\pi\varepsilon_0} \cdot \frac{-x}{\|\vec{r}\|^3}$$

$$\frac{\partial U}{\partial y}(\vec{r}) = -\frac{Q}{4\pi\varepsilon_0} \cdot \frac{y}{\|\vec{r}\|^3}, \quad \frac{\partial U}{\partial z}(\vec{r}) = -\frac{Q}{4\pi\varepsilon_0} \cdot \frac{z}{\|\vec{r}\|^3}$$

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\|\vec{r}\|^3} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boxed{\frac{Q}{4\pi\varepsilon_0} \frac{\vec{r}}{\|\vec{r}\|^3}} = \frac{Q}{4\pi\varepsilon_0} \frac{\vec{e}_r}{\|\vec{r}\|^2}$$

where $\vec{e}_r$ is the unit vector in the direction of $\vec{r}$.