The Bright Side of Mathematics

Linear Algebra

Eigenvalues and eigenvectors

Exercise 1. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{bmatrix} 5 & 6 \\ 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 6 \\ 0 & 5 \end{bmatrix}$$

$Av = \lambda v \qquad v \neq 0$

$(A - I\lambda)v = \vec{0} \qquad \det(A - I\lambda) = 0$

$$\begin{vmatrix} 5-\lambda & 6 \\ 0 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 = 0 \qquad \boxed{\lambda = 5}$$

$$\begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 6x_2 = 0, \quad x_2 = 0 \qquad \boxed{v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t, \ t \in \mathbb{R}\setminus\{0\}}$$

$$B = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} -2-\lambda & -8 & -12 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = +(1-\lambda)\left[(-2-\lambda)(4-\lambda) + 8\right]$$

$$= (1-\lambda)\left[-8 + 2\lambda - 4\lambda + \lambda^2 + 8\right] = (1-\lambda)(-2\lambda + \lambda^2)$$

$$= \lambda(1-\lambda)(\lambda-2) = 0$$

$$\boxed{\lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = 2}$$

$\lambda_1 = 0$

$$\begin{pmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -4 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 4 \\ 0 & 0 & -4 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = -4x_2$

$x_1 + 4x_2 + 4x_3 = 0$

$x_3 = 0$

$$\boxed{v_1 = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} t, \ t \in \mathbb{R}\setminus\{0\}}$$

$\lambda_2 = 1$

$$\begin{pmatrix} -3 & -8 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + 3x_2 + 4x_3 = 0 \to x_1 = -4x_3$

$x_2 = 0$

$$\boxed{v_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \cdot t, \ t \in \mathbb{R}\setminus\{0\}}$$

$\lambda_3 = 2$

$$B = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} -4 & -8 & -12 \\ 1 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x_3 = 0$

$x_1 + 2x_2 + 4x_3 = 0$

$x_1 = -2x_2$

$$\boxed{v_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} t, \ t \in \mathbb{R}\setminus\{0\}}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ -1 & 1 & 1-\lambda & 1 \\ -1 & 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} =$$

$$(1-\lambda)\left[-\lambda(1-\lambda)(2-\lambda) + 1(1-\lambda)\right] = (1-\lambda)^2\left[-\lambda(2-\lambda) + 1\right]$$

$$(1-\lambda)^2(-2\lambda + \lambda^2 + 1) = (1-\lambda)^2(1-\lambda)^2 = 0 \qquad \boxed{\lambda = 1}$$

$$\begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x_1 + x_2 + x_4 = 0$

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{V = \{c_1 v_1 + c_2 v_2 + c_3 v_3, \ c_1, c_2, c_3 \in \mathbb{R}\} \setminus \{\vec{0}\}}$$

$$A(\underbrace{c_1 v_1 + c_2 v_2 + c_3 v_3}_{\vec{v}}) = c_1 A v_1 + c_2 A v_2 + c_3 A v_3 =$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 1 \cdot \vec{v}$$