Fourier Series

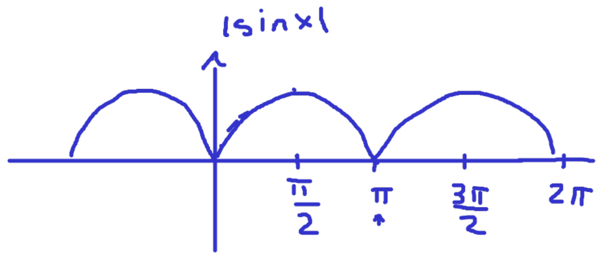
Exercises 1

Exercise 1. Compute the Fourier series of $f(x) = |\sin(x)|$.

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega x) + b_k \sin(k\omega x) \right) \qquad \omega = \frac{2\pi}{T}$$

$$a_k = \frac{2}{T} \int_0^T f(x) \cos(k\omega x)\, dx \quad , k \geq 0$$

$$b_k = \frac{2}{T} \int_0^T f(x) \sin(k\omega x)\, dx \quad , k \geq 1$$



$\rightarrow$ even : $b_k = 0$

$T = \pi \qquad \omega = \frac{2\pi}{\pi} = 2$

$$\frac{a_0}{2} = \frac{1}{\pi} \int_0^\pi \sin(x)\, dx = \frac{1}{\pi}\left(-\overset{-1}{\cos \pi} + \overset{1}{\cos 0}\right) = \frac{2}{\pi}$$

$$a_k = \frac{2}{\pi} \int_0^\pi \sin(x) \cos(2kx)\, dx$$

$$\int \sin(x) \cos(2kx)\, dx = -\cos(x)\cos(2kx) - 2k \int \cos(x) \sin(2kx)\, dx$$

$f'(x) = \sin(x) \qquad g(x) = \cos(2kx)$

$f(x) = -\cos(x) \qquad g'(x) = -\sin(2kx)\, 2k$

$f'(x) = \cos x \qquad g(x) = \sin(2kx)$

$f(x) = \sin x \qquad g'(x) = \cos(2kx)\, 2k$

$$\int \sin(x)\cos(2kx)\, dx = -\cos(x)\cos(2kx) - 2k\left(\sin(x)\sin(2kx) - 2k \int \sin x \cos(2kx)\, dx\right)$$

$$(1-4k^2)\int_0^\pi \sin(x)\cos(2kx)\, dx = \left(-\cos(x)\cos(2kx) - 2k\sin(x)\sin(2kx)\right)\Big|_0^\pi$$

$$\int_0^\pi \sin(x)\cos(2kx)\, dx = \frac{1}{1-4k^2}\left(\underbrace{-(-1)(1)}_{1} \underbrace{-\left(-(1)(1)\right)}_{+\ 1}\right) = \frac{2}{1-4k^2}$$

$$a_k = \frac{2}{\pi} \cdot \frac{2}{1-4k^2} \quad k \geq 1 \quad , \quad \frac{a_0}{2} = \frac{2}{\pi} \quad , \quad b_k = 0$$

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega x) + b_k \sin(k\omega x) \right)$$

$$\boxed{|\sin(x)| \approx \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4}{\pi(1-4k^2)} \cos(2kx)}$$