The Bright Side of Mathematics

Distributions – Part 12

$T \in \mathcal{D}'(\mathbb{R}^n)$, $K \subseteq \mathbb{R}^n$ (compact). There is $m \in \mathbb{N}_0$, $c > 0$ such that:

$$\text{supp}(\varphi) \subseteq K \implies |\langle T, \varphi \rangle| \leq c \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty$$

$$\leq \tilde{c} \cdot \underbrace{\max\left\{ |D^\alpha \varphi(x)| \;\middle|\; x \in \mathbb{R}^n, |\alpha| \leq m \right\}}_{\|\varphi\|_m}$$

Definition: $T \in \mathcal{D}'(\mathbb{R}^n)$ is called a distribution of finite order m if:

$$\underset{m \in \mathbb{N}_0}{\exists} \quad \underset{\substack{K \subseteq \mathbb{R}^n \\ \text{compact}}}{\forall} \quad \underset{c > 0}{\exists} \quad \underset{\varphi \in \mathcal{D}(\mathbb{R}^n)}{\forall} \quad \text{supp}(\varphi) \subseteq K \implies |\langle T, \varphi \rangle| \leq c \cdot \|\varphi\|_m$$

Regular distribution: $$|\langle T_f, \varphi \rangle| = \left| \int_K f(x)\, \varphi(x)\, dx \right| \leq \underbrace{\int_K |f(x)|\, dx}_{c} \cdot \|\varphi\|_0$$

$\implies$ of order 0

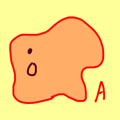
Theorem: $$\{ T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C} \mid T \text{ is of order } 0 \}$$

$\overset{\text{bijection}}{\longleftrightarrow}$

$$\{ \mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{C} \cup \{\infty\} \mid \mu \text{ complex Radon measure} \}$$

For μ define: $\langle T_\mu, \varphi \rangle := \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x)$

Example: Dirac measure: $$\delta(A) := \begin{cases} 0 , & 0 \notin A \\ 1 , & 0 \in A \end{cases}$$



Corresponding distribution: $\langle T_\delta, \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x)\, d\delta(x) = \varphi(0)$

$\implies T_\delta$ is the delta distribution