Distributions – Part 11

$$T \in \mathcal{D}'(\mathbb{R}^n) \implies D^\alpha T \in \mathcal{D}'(\mathbb{R}^n) \quad \text{(for any multi-index } \alpha\text{)}$$

Therefore: $D^\alpha : \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

- linear
- continuous

Result: For distributions T_k $(k \in \mathbb{N})$, we have:

$$D^\alpha\left(\sum_{k=1}^{\infty} T_k\right) = \sum_{k=1}^{\infty} D^\alpha T_k$$

Example: Laplace's equation: $\Delta T = 0$ $\quad \left(\Delta = D^\alpha + D^\beta + D^\gamma \right)$, $\alpha = (2,0,0)$, $\beta = (0,2,0)$, $\gamma = (0,0,2)$

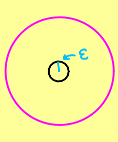
$n = 3$

$$\gamma(x) = -\frac{1}{4\pi} \cdot \frac{1}{\|x\|}$$

$\|x\|$: euclidean/standard norm in $\mathbb{R}^3$

regular distribution: T_γ, $\quad \gamma \in \mathcal{L}^1_{loc}(\mathbb{R}^3)$

$$\langle \Delta T_\gamma, \varphi \rangle = \langle T_\gamma, \Delta\varphi \rangle = \int_{B_R(0)} \gamma(x)\, \Delta\varphi(x)\, dx$$



$$= \int_{B_R(0)\setminus B_\varepsilon(0)} \gamma(x)\, \Delta\varphi(x)\, dx + \int_{B_\varepsilon(0)} \gamma(x)\, \Delta\varphi(x)\, dx$$

$$= \lim_{\varepsilon \to 0^+} \int_{B_R(0)\setminus B_\varepsilon(0)} \gamma(x)\, \Delta\varphi(x)\, dx$$

Use Green's identities!

$$= \varphi(0) = \langle \delta, \varphi \rangle$$

$$\implies \Delta T_\gamma = \delta \quad \text{(fundamental solution)}$$

Definition: For a differential operator $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$, $\quad (P(D) = 0)$

we call $T \in \mathcal{D}'(\mathbb{R}^n)$ a fundamental solution if

$$P(D)\, T = \delta$$