The Bright Side of Mathematics

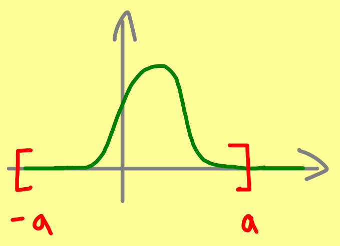
# Distributions – Part 10

Motivation: $f \in C^1(\mathbb{R}^n)$ $(n = 1)$

We get two regular distributions: $T_f \,,\; T_{f'} \in \mathcal{D}'(\mathbb{R}^n)$

We have: $$\langle T_{f'}, \varphi \rangle = \int_{\mathbb{R}} f'(x)\, \varphi(x)\, dx$$

$\mathbb{R} \leftarrow [-a,a] \supseteq \operatorname{supp}(\varphi)$




$$= \int_{-a}^{a} f'(x)\, \varphi(x)\, dx$$

$$= \underbrace{f(x)\, \varphi(x) \Big|_{-a}^{a}}_{=0} - \int_{-a}^{a} f(x)\, \varphi'(x)\, dx$$

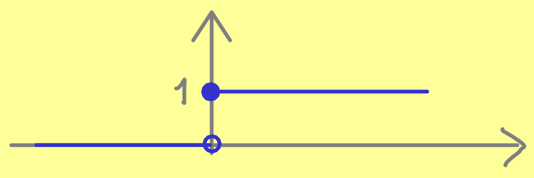
$$= \langle -T_f , \varphi' \rangle$$

Definition: For a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$, we define a new distribution $D^\alpha T \in \mathcal{D}'(\mathbb{R}^n)$ (for any multi-index $\alpha$), called the (distributional) partial derivative of $T$, by: $$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$$

Note: $D^\alpha(T_f) = T_{D^\alpha f}$ for $f \in C^\infty(\mathbb{R}^n)$

Example: (a) Heaviside function $H : \mathbb{R} \longrightarrow \mathbb{R}$ $(n = 1)$

$\alpha = (1)$




$$\langle D^\alpha (T_H), \varphi \rangle = (-1)^1 \langle T_H , \varphi' \rangle$$

$$= - \int_{\mathbb{R}} H(x)\, \varphi'(x)\, dx = - \int_{-a}^{a} H(x)\, \varphi'(x)\, dx$$

$\mathbb{R} \leftarrow [-a,a] \supseteq \operatorname{supp}(\varphi') \supseteq \operatorname{supp}(\varphi)$

$$= - \int_{0}^{a} 1 \cdot \varphi'(x)\, dx = - \int_{0}^{a} \varphi'(x)\, dx$$

$$= \underbrace{-\varphi(a)}_{=0} + \varphi(0) = \langle \delta , \varphi \rangle$$

distributional derivative of $H$ $= \delta$

(b) $n = 1$, $\alpha = (1)$ $$\langle D^\alpha \delta, \varphi \rangle = - \langle \delta , \varphi' \rangle = - \varphi'(0)$$

distributional derivative of $\delta$