The Bright Side of Mathematics

Distributions – Part 8

$T, S \in \mathcal{D}'(\mathbb{R}^n) \rightsquigarrow T \cdot S$ makes problems...

Multiplication with smooth functions: $S \in \mathcal{D}'(\mathbb{R}^n)$, $f \in C^\infty(\mathbb{R}^n)$

$T_f \cdot S$ can be defined as a new distribution.

First case: S is a regular distribution, $S = T_g$ with $g \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$

$$(T_f \cdot T_g)(\varphi) \overset{\text{should be}}{=} T_{f\cdot g}(\varphi) = \int_{\mathbb{R}^n} (f(x)\cdot g(x))\,\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n} g(x)\big(f(x)\varphi(x)\big)dx = T_g(f\cdot\varphi) \quad \text{with } f\cdot\varphi \in \mathcal{D}(\mathbb{R}^n)$$

Definition: $T_f \cdot S$ or $f \cdot S$ for $f \in C^\infty(\mathbb{R}^n)$ is the distribution defined by:

$$\langle f\cdot S, \varphi\rangle := \langle S, f\cdot\varphi\rangle \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n)$$

Proof: (1) $f\cdot S : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is linear ✓

(2) Leibniz rule: $D^\alpha(f\cdot\varphi) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (D^\beta f)\cdot(D^{\alpha-\beta}\varphi)$

$$\boxed{S \text{ is a distribution} \iff \underset{\substack{K\subseteq\mathbb{R}^n\\ \text{compact}}}{\forall}\ \underset{m\in\mathbb{N}_0}{\exists}\ \underset{C>0}{\exists}\ \underset{\tilde\varphi\in\mathcal{D}(\mathbb{R}^n)}{\forall} \quad \operatorname{supp}(\tilde\varphi)\subseteq K \Rightarrow |S(\tilde\varphi)| \leq C\cdot\sum_{|\alpha|\leq m}\|D^\alpha\tilde\varphi\|_\infty}$$

$$|(f\cdot S)(\varphi)| = |S(f\varphi)| \leq C\cdot\sum_{|\alpha|\leq m}\|D^\alpha(f\varphi)\|_\infty$$

$$\leq C\cdot\sum_{|\alpha|\leq m}\sum_{\beta\leq\alpha}\|(D^\beta f)\|_\infty\,\|(D^{\alpha-\beta}\varphi)\|_\infty\binom{\alpha}{\beta}$$

$$\leq \tilde C\cdot\sum_{|\alpha|\leq m}\|D^\alpha\varphi\|_\infty$$