Distributions - Part 4

Space of distributions

$\mathcal{D}(\mathbb{R}^n)$ – vector space of test functions with notion of convergence for sequences of test functions (special case of a topological vector space)

$T: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is called distribution if

- T is linear $\left(\begin{array}{l} T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2) \\ T(\lambda \cdot \varphi) = \lambda \cdot T(\varphi) \end{array} \right)$
- T is continuous in the following sense:

  For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and all sequences $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$ with $\varphi_k \xrightarrow[\mathcal{D}]{} \varphi$:

  $$T(\varphi_k) \xrightarrow{k \to \infty} T(\varphi) \quad \left(\text{sequentially continuous}\right)$$

Notation: $\mathcal{D}(\mathbb{R}^n)'$ or $\mathcal{D}'(\mathbb{R}^n)$ space of distributions

important!

elements: $T: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$

Examples: (a) delta distribution: $\delta: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$

$$\delta(\varphi) = \varphi(0)$$

- linear ✓
- continuous: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$ with $\varphi_k \xrightarrow[\mathcal{D}]{} \varphi$ $\left(\text{in particular: } \varphi_k(x) \xrightarrow{k \to \infty} \varphi(x) \text{ for all } x \in \mathbb{R}^n\right)$,

  We have: $\delta(\varphi_k) = \varphi_k(0) \xrightarrow{k \to \infty} \varphi(0) = \delta(\varphi)$ ✓

(b) Continuous functions "are" distributions:

For $f \in C(\mathbb{R}^n)$ define: $T_f: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x) \varphi(x) \, dx$$

- linear ✓
- continuous: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$ with $\varphi_k \xrightarrow[\mathcal{D}]{} \varphi$ $\left(\text{in particular: } \varphi_k \xrightarrow[\text{uniform convergence}]{} \varphi\right)$,

  We have $T_f(\varphi_k) \xrightarrow{k \to \infty} T_f(\varphi)$ ✓

Important property: $f, g \in C(\mathbb{R}^n)$, $f \neq g \implies T_f \neq T_g$