The Bright Side of Mathematics

Distributions – Part 3

Convergence for test functions

$\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ (space of test functions)

Notions of convergence:

- If one has a norm $\|\cdot\|$:

$$f_n \longrightarrow f \quad :\Longleftrightarrow \quad \|f_n - f\| \xrightarrow{n\to\infty} 0$$

- If one has a metric $d(\cdot,\cdot)$:

$$f_n \longrightarrow f \quad :\Longleftrightarrow \quad d(f_n, f) \xrightarrow{n\to\infty} 0$$

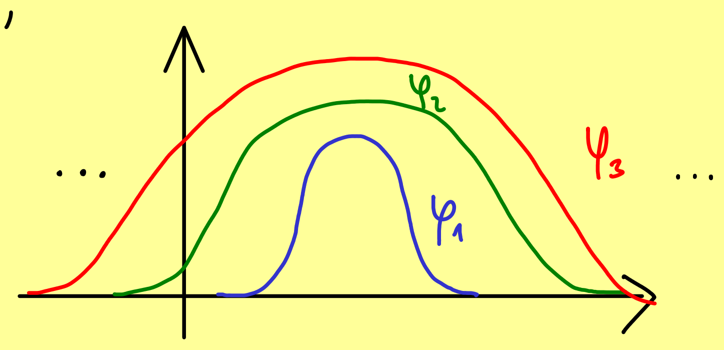
$\Longrightarrow$ Both notions are not sufficient here $\rightsquigarrow$ very strong notion needed

Definition: For $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we write

$$\varphi_k \xrightarrow[\mathcal{D}]{} \varphi \quad \text{if}$$

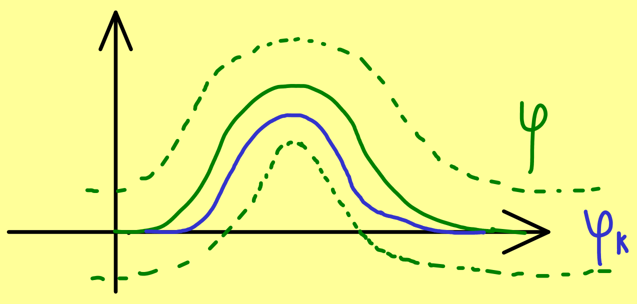
(a) There is a bounded set M, such that outside of it: $\varphi_k = 0$ for all k.

Not:



(b) Uniform convergence $\varphi_k \xrightarrow[\text{unif.}]{} \varphi$ and for all multi-indices α

$$D^\alpha \varphi_k \xrightarrow[\text{unif.}]{} D^\alpha \varphi$$



Use the supremum norm $\|f\|_\infty := \sup\{ |f(x)| \mid x \in \mathbb{R}^n \}$.

$\varphi_k \xrightarrow[\mathcal{D}]{} \varphi \quad \Longleftrightarrow \quad$ (a) $\exists\, C \subseteq \mathbb{R}^n$ compact such that $\operatorname{supp}(\varphi_1), \operatorname{supp}(\varphi_2), \ldots \subseteq C$

(b) $\forall \alpha$ multi-index we have $\|D^\alpha \varphi_k - D^\alpha \varphi\|_\infty \xrightarrow{k\to\infty} 0$