Distributions - Part 7

$f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ $\longleftarrow$ vector space of functions

$\rightsquigarrow$ $\qquad$ $\rightsquigarrow$

$T_f \in \mathcal{D}'(\mathbb{R}^n)$ $\longleftarrow$ vector space of distributions?

Fact: $\mathcal{D}'(\mathbb{R}^n)$ is a real (or complex) vector space:

- addition: $+$ for $T, S \in \mathcal{D}'(\mathbb{R}^n)$, define $T + S \in \mathcal{D}'(\mathbb{R}^n)$

$$(T+S)(\varphi) = T(\varphi) + S(\varphi)$$

$$(T_f + T_g)(\varphi) = T_f(\varphi) + T_g(\varphi)$$

$$= \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx + \int_{\mathbb{R}^n} g(x)\,\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n} \big(f(x) + g(x)\big)\varphi(x)\,dx = T_{f+g}(\varphi)$$

- scalar multiplication: $\cdot$ for $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $T \in \mathcal{D}'(\mathbb{R}^n)$

  define $\lambda \cdot T$ by:

$$(\lambda \cdot T)(\varphi) = \lambda \cdot T(\varphi)$$

(we have all calculations rules in a vector space)

duality pairing: $\langle T, \varphi \rangle := T(\varphi)$

$$\langle \cdot , \cdot \rangle : \mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R} \text{ (or } \mathbb{C}) \qquad \text{bilinear map}$$