

Distributions - part 6

Delta distribution is not regular:

There is no locally integrable function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ (or \mathbb{C})

with $\delta(\varphi) = T_f(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$
 $\varphi(0)$ $\stackrel{!!}{=} \int_{\mathbb{R}^n} f(x) \varphi(x) dx$

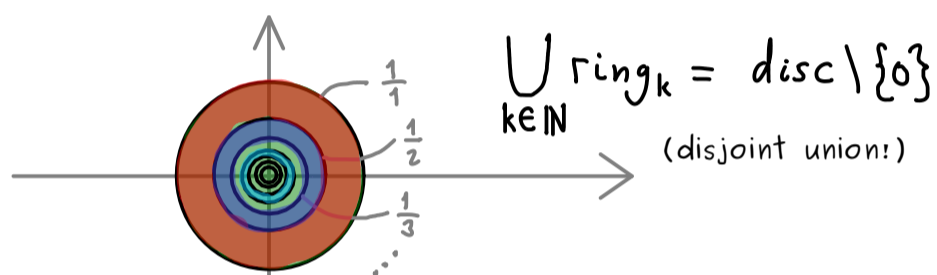
Proof: Assume there is $f \in \mathcal{L}'_{loc}(\mathbb{R}^n)$ with $\varphi(0) = \int_{\mathbb{R}^n} f(x) \varphi(x) dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

① $\int_{\|x\| \leq 1} |f(x)| dx = a < \infty$

$\int_{\bigcup_{k \in \mathbb{N}} \text{ring}_k} |f(x)| dx$

$\stackrel{!!}{=} \text{(measure theory / integration theory)}$

$\sum_{k=1}^{\infty} \int_{\text{ring}_k} |f(x)| dx \Rightarrow \exists k_0 \in \mathbb{N} : \sum_{k=k_0}^{\infty} \int_{\text{ring}_k} |f(x)| dx \leq \frac{1}{2}$



In summary: There is $\varepsilon > 0$ with $\int_{\|x\| \leq \varepsilon} |f(x)| dx = b \leq \frac{1}{2}$

② Take test function: $\varphi_\varepsilon(x) = \begin{cases} 0 & , \|x\| \geq \varepsilon \\ \exp\left(-\frac{1}{1 - \left(\frac{\|x\|}{\varepsilon}\right)^2}\right) & , \|x\| < \varepsilon \end{cases}$

$\varphi_\varepsilon(0) = \left| \int_{\mathbb{R}^n} f(x) \varphi_\varepsilon(x) dx \right| \leq \int_{\|x\| \leq \varepsilon} |f(x)| |\varphi_\varepsilon(x)| dx \leq \underbrace{\|\varphi_\varepsilon\|_\infty}_{\varphi_\varepsilon(0)} \cdot \underbrace{\int_{\|x\| \leq \varepsilon} |f(x)| dx}_b$
 $\leq \varphi_\varepsilon(0) \cdot \frac{1}{2} \Rightarrow \text{contradiction}$