Distributions – part 6

Delta distribution is not regular:

There is no locally integrable function $f: \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$)

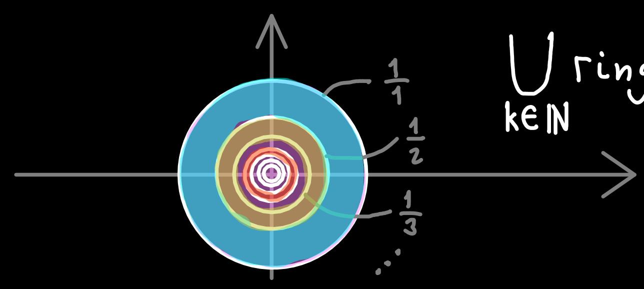
with $\quad \underbrace{\delta(\varphi)}_{= \varphi(0)} = \underbrace{T_f(\varphi)}_{= \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx} \quad$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

Proof: Assume there is $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ with $\varphi(0) = \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

①

$$\int_{\|x\| \leq 1} |f(x)|\,dx = a < \infty$$

$$\int_{\|x\| \leq 1} |f(x)|\,dx = \int_{\bigcup_{k \in \mathbb{N}} \text{ring}_k} |f(x)|\,dx$$



$\bigcup_{k \in \mathbb{N}} \text{ring}_k = \text{disc} \setminus \{0\}$

(disjoint union!)

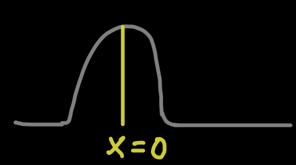
(measure theory/ integration theory)

$$= \sum_{k=1}^{\infty} \int_{\text{ring}_k} |f(x)|\,dx \implies \exists k_0 \in \mathbb{N} : \sum_{k=k_0}^{\infty} \int_{\text{ring}_k} |f(x)|\,dx \leq \frac{1}{2}$$

In summary: There is $\varepsilon > 0$ with $\displaystyle\int_{\|x\| \leq \varepsilon} |f(x)|\,dx = b \leq \frac{1}{2}$

②

Take test function: $\quad \varphi_\varepsilon(x) = \begin{cases} 0 & , \ \|x\| \geq \varepsilon \\ \exp\left(-\frac{1}{1 - \left(\frac{\|x\|}{\varepsilon}\right)^2}\right) & , \ \|x\| < \varepsilon \end{cases}$



$$\varphi_\varepsilon(0) = \left| \int_{\mathbb{R}^n} f(x)\,\varphi_\varepsilon(x)\,dx \right| \leq \int_{\|x\| \leq \varepsilon} |f(x)| \cdot |\varphi_\varepsilon(x)|\,dx \leq \underbrace{\|\varphi_\varepsilon\|_\infty}_{\varphi_\varepsilon(0)} \cdot \underbrace{\int_{\|x\| \leq \varepsilon} |f(x)|\,dx}_{b}$$

$$\leq \varphi_\varepsilon(0) \cdot \frac{1}{2} \implies \text{contradiction}$$