BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

Distributions – part 6

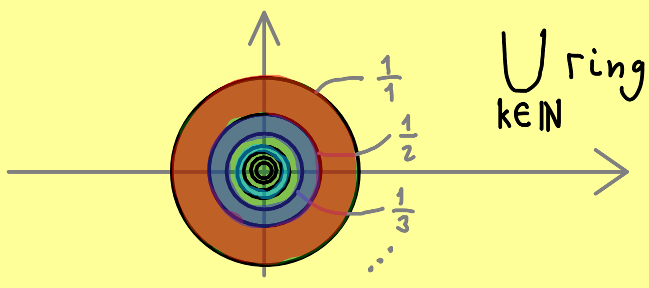
Delta distribution is not regular:

There is no locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$)

with $\underset{\varphi(0)}{\underset{\parallel}{\delta(\varphi)}} = T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

Proof: Assume there is $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ with $\varphi(0) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(1) $\int_{\|x\|\leq 1} |f(x)|\,dx = a < \infty$



$\bigcup_{k\in\mathbb{N}} \mathrm{ring}_k = \mathrm{disc}\setminus\{0\}$ (disjoint union!)

$= \int_{\bigcup_{k\in\mathbb{N}} \mathrm{ring}_k} |f(x)|\,dx$

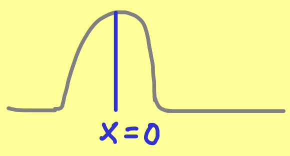
$=$ (measure theory/ integration theory)

$\sum_{k=1}^{\infty} \int_{\mathrm{ring}_k} |f(x)|\,dx \implies \exists k_0 \in \mathbb{N} : \sum_{k=k_0}^{\infty} \int_{\mathrm{ring}_k} |f(x)|\,dx \leq \frac{1}{2}$

In summary: There is $\varepsilon > 0$ with $\int_{\|x\|\leq\varepsilon} |f(x)|\,dx = b \leq \frac{1}{2}$

(2) Take test function:

$$\varphi_\varepsilon(x) = \begin{cases} 0 & , \|x\| \geq \varepsilon \\ \exp\left(-\frac{1}{1-\left(\frac{\|x\|}{\varepsilon}\right)^2}\right) & , \|x\| < \varepsilon \end{cases}$$



$$\varphi_\varepsilon(0) = \left| \int_{\mathbb{R}^n} f(x)\varphi_\varepsilon(x)\,dx \right| \leq \int_{\|x\|\leq\varepsilon} |f(x)|\cdot|\varphi_\varepsilon(x)|\,dx \leq \underbrace{\|\varphi_\varepsilon\|_\infty}_{\varphi_\varepsilon(0)} \cdot \underbrace{\int_{\|x\|\leq\varepsilon} |f(x)|\,dx}_{b}$$

$$\leq \varphi_\varepsilon(0)\cdot\frac{1}{2} \implies \text{contradiction}$$