

Distributions - Part 3

Convergence for test functions

$$\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n) \quad (\text{space of test functions})$$

Notions of convergence:

• If one has a norm $\|\cdot\|$:

$$f_n \rightarrow f \quad :\Leftrightarrow \quad \|f_n - f\| \xrightarrow{n \rightarrow \infty} 0$$

• If one has a metric $d(\cdot, \cdot)$:

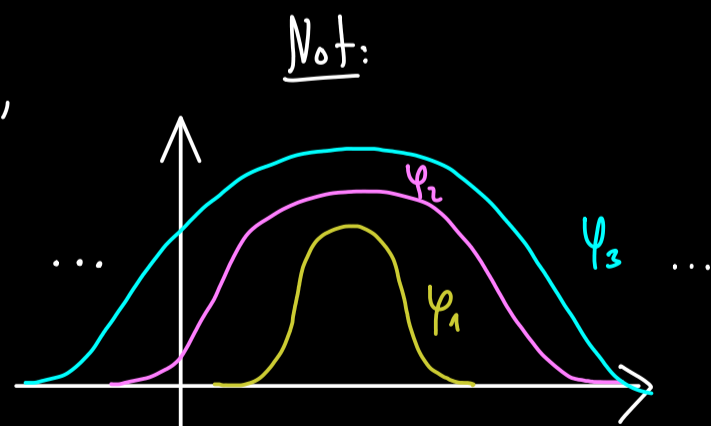
$$f_n \rightarrow f \quad :\Leftrightarrow \quad d(f_n, f) \xrightarrow{n \rightarrow \infty} 0$$

\Rightarrow Both notions are not sufficient here \rightsquigarrow very strong notion needed

Definition: For $(\varphi_k)_{k \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we write

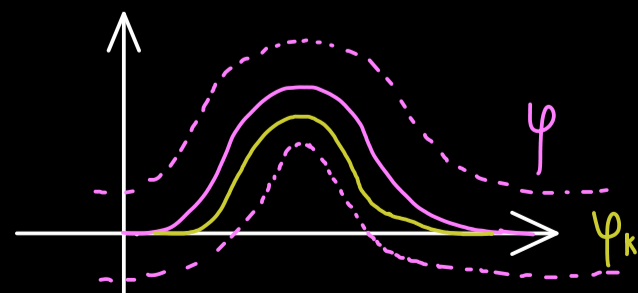
$$\varphi_k \xrightarrow{\mathcal{D}} \varphi \quad \text{if}$$

- (a) There is a bounded set M ,
such that outside of it:
 $\varphi_k = 0$ for all k .



- (b) Uniform convergence $\varphi_k \xrightarrow{\text{unif.}} \varphi$
and for all multi-indices α

$$D^\alpha \varphi_k \xrightarrow{\text{unif.}} D^\alpha \varphi$$



Use the supremum norm $\|f\|_\infty := \sup \{ |f(x)| \mid x \in \mathbb{R}^n \}$.

$$\psi_k \xrightarrow{\mathcal{D}} \psi \quad \Leftrightarrow \quad \begin{aligned} & \text{(a) } \exists C \subseteq \mathbb{R}^n \text{ compact such that} \\ & \quad \text{supp}(\psi_1), \text{supp}(\psi_2), \dots \subseteq C \\ & \text{(b) } \forall \alpha \text{ multi-index we have} \\ & \quad \|D^\alpha \psi_k - D^\alpha \psi\|_\infty \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$