The Bright Side of Mathematics

Distributions - Part 17

Convolution from part 13: $\ast : \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

defined by: $\langle \psi \ast T, \varphi \rangle = \langle T, \check{\psi} \ast \varphi \rangle$

where $\check{\psi}(x) := \psi(-x)$

Convolution (extended): $\ast : \mathcal{D}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

Definition: For $S \in \mathcal{E}'(\mathbb{R}^n)$, we define a new distribution:

$$\langle \check{S}, \varphi \rangle = \langle S, \check{\varphi} \rangle$$

Easy to show: $\left(\langle T_{\check{f}}, \varphi \rangle = \langle T_f, \check{\varphi} \rangle \right)$

We get: $\check{S} \in \mathcal{E}'(\mathbb{R}^n)$.

Proposition: For $\psi \in \mathcal{D}(\mathbb{R}^n)$, $S \in \mathcal{E}'(\mathbb{R}^n)$, we get:

$\psi \ast \check{S}$ is a regular distribution

and $\psi \ast \check{S} \in C^\infty(\mathbb{R}^n)$

and $\psi \ast \check{S} \in \mathcal{D}(\mathbb{R}^n)$ $\quad$ ($\mathrm{supp}(\psi \ast \check{S})$ compact)

Definition: The convolution $\ast : \mathcal{D}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

is given by $\langle T \ast S, \varphi \rangle := \langle T, \varphi \ast \check{S} \rangle$

Compatible to old definition: Choose regular distribution $T = T_\psi$ with $\psi \in \mathcal{D}(\mathbb{R}^n)$

$$\langle T_\psi \ast S, \varphi \rangle = \langle T_\psi, \underbrace{\varphi \ast \check{S}}_{= T_g} \rangle = \int_{\mathbb{R}^n} \psi(x)\, g(x)\, dx$$

$$= \langle \varphi \ast \check{S}, \psi \rangle = \langle \check{S}, \check{\varphi} \ast \psi \rangle$$

$$= \langle S, (\check{\varphi} \ast \psi)^\vee \rangle = \langle S, \check{\psi} \ast \varphi \rangle \quad \checkmark$$

Important property:

$T \ast \delta = T$ $\quad$ for all $T \in \mathcal{D}'(\mathbb{R}^n)$

$\delta \ast S = S$ $\quad$ for all $S \in \mathcal{E}'(\mathbb{R}^n)$