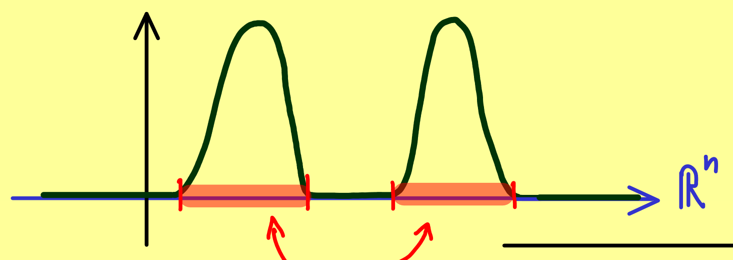
The Bright Side of Mathematics

Distributions - Part 15

Support: $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

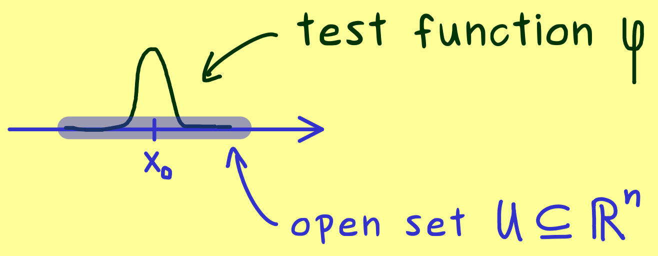


$\mathrm{supp}(f) := \overline{\{ x \in \mathbb{R}^n \mid f(x) \neq 0 \}}$

complement is the largest open set $U \subseteq \mathbb{R}^n$ such that $f|_U = 0$

Local behaviour of a distribution? $T \in \mathcal{D}'(\mathbb{R}^n)$ What is the value of T at a point $x_0 \in \mathbb{R}^n$?

$\hookrightarrow$ not meaningful



$$T = 0 \text{ in } U \; :\Longleftrightarrow \; T(\varphi) = 0 \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^n) \text{ with } \mathrm{supp}(\varphi) \subseteq U$$

Example: $\delta \in \mathcal{D}'(\mathbb{R}^n)$: $\delta = 0$ in $\mathbb{R}^n \setminus \{0\}$ (since $\delta(\varphi) = \varphi(0)$)

$\hookrightarrow$ support of δ is given by $\{0\}$

Proposition: For $T \in \mathcal{D}'(\mathbb{R}^n)$, there a maximal open set $U_{max} \subseteq \mathbb{R}^n$ with

$$T = 0 \text{ in } U_{max} .$$

The complement is called the support of T :

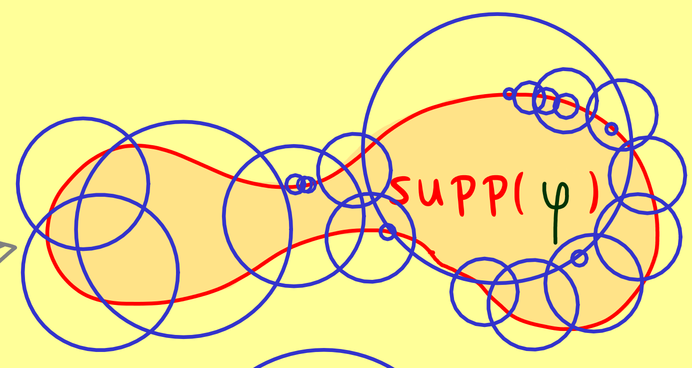
$$\mathrm{supp}(T) := \mathbb{R}^n \setminus U_{max} \quad \text{(closed set)}$$

Proof: Define: $\mathcal{U} := \{ U \subseteq \mathbb{R}^n \text{ open} \mid T = 0 \text{ in } U \}$

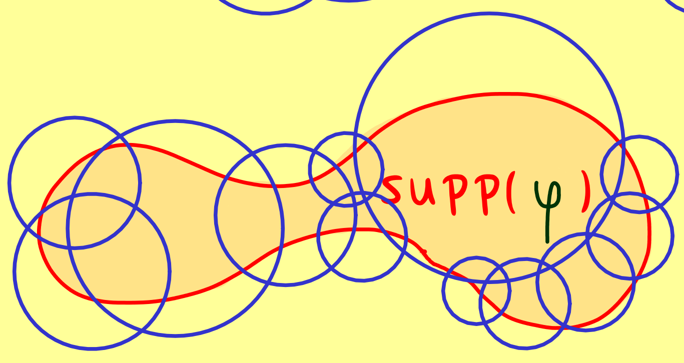
$U_{max} := \bigcup_{U \in \mathcal{U}} U$ Question: Do we have $T = 0$ in U_{max} ?

Take $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\mathrm{supp}(\varphi) \subseteq U_{max}$

$\bigcup_{U \in \mathcal{U}} U$ covering



$\mathrm{supp}(\varphi)$ is compact $\Longrightarrow$ we have a finite subcover



$$\mathrm{supp}(\varphi) \subseteq U_1 \cup U_2 \cup \cdots \cup U_m$$

> **Partition of unity:** There are test functions $\psi_1, \psi_2, \ldots, \psi_m \in \mathcal{D}(\mathbb{R}^n)$
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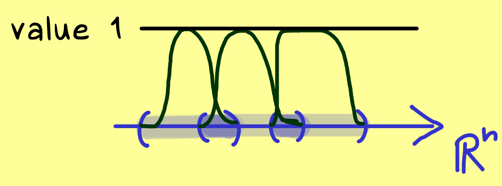
> 
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> with $\mathrm{supp}(\psi_i) \subseteq U_i$ such that:
>
> $$1 = \sum_{i=1}^m \psi_i(x) \quad \text{for all } x \in \mathrm{supp}(\varphi)$$

partition of unity $\Longrightarrow$ $\varphi(x) = \sum_{i=1}^m \psi_i(x) \cdot \varphi(x)$ for all $x \in \mathbb{R}^n$ $\quad \mathrm{supp}(\psi_i \varphi) \subseteq U_i$

$$\Longrightarrow \langle T, \varphi \rangle = \langle T, \sum_{i=1}^m \psi_i \varphi \rangle = \sum_{i=1}^m \underbrace{\langle T, \psi_i \varphi \rangle}_{=0} = 0 \qquad \square$$