# Distributions – Part 14

convolution: $* : \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

$$\langle \psi * \delta , \varphi \rangle = \langle \delta , \check{\psi} * \varphi \rangle \qquad \text{with } \check{\psi}(z) = \psi(-z)$$

$$= (\check{\psi} * \varphi)(0)$$

$$= \int_{\mathbb{R}^n} \check{\psi}(0-y)\, \varphi(y)\, dy$$

$$= \int_{\mathbb{R}^n} \psi(y)\, \varphi(y)\, dy = \langle T_\psi , \varphi \rangle$$

Hence: $\psi * \delta = \psi$ for all $\psi \in \mathcal{D}(\mathbb{R}^n)$

($\psi$: seen as a regular distribution)

($\delta$: neutral element for $*$)

Properties: (a) For all multi-indices $\alpha$ :

$$D^\alpha(\psi * T) = (D^\alpha \psi) * T = \psi * (D^\alpha T)$$

(b) $\psi_1 * (\psi_2 * T) = (\psi_1 * \psi_2) * T$

Application: differential operator: $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$

fundamental solution: $P(D) E = \delta , \quad E \in \mathcal{D}'(\mathbb{R}^n)$

partial differential equation: $P(D) u = f$ ⟶ search for $u$

$(\Delta u = f)$

How about $u = f * E$ ?

$$P(D) u = P(D)(f * E) = f * (\underbrace{P(D) E}_{\delta}) = f$$